

# PROJECTIVE MODELS OF THE SUPERSINGULAR $K3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 5

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**ABSTRACT.** Let  $X$  be a supersingular  $K3$  surface in characteristic 5 with Artin invariant 1, which is unique up to isomorphisms. Then  $X$  has a polarization of degree 2 that realizes  $X$  as the double cover of the projective plane branching along the Fermat sextic curve. We present a list of polarizations of  $X$  with degree 2 whose intersection number with this Fermat sextic polarization is less than or equal to 5, and give the defining equations of the corresponding projective models. We also present a method to describe birational morphisms between these projective models explicitly. As a by-product, a non-projective automorphism of the Fermat sextic double plane is obtained.

## 1. INTRODUCTION

A  $K3$  surface defined over an algebraically closed field is said to be *supersingular* (in the sense of Shioda) if its Néron-Severi lattice is of rank 22. Supersingular  $K3$  surfaces exist only in positive characteristics. Let  $Y$  be a supersingular  $K3$  surface in characteristic  $p > 0$ . Artin [3] showed that the discriminant of the Néron-Severi lattice  $\text{NS}(Y)$  is written as  $-p^{2\sigma}$ , where  $\sigma$  is an integer such that  $1 \leq \sigma \leq 10$ . This integer  $\sigma$  is called the *Artin invariant* of  $Y$ . It is proved in [12, 13, 15] that, for each prime  $p$ , a supersingular  $K3$  surface with Artin invariant 1 in characteristic  $p$  exists and is unique up to isomorphisms. Several detailed study of supersingular  $K3$  surfaces with Artin invariant 1 in small characteristics have appeared recently (see [5, 7, 8, 10]).

The purpose of this paper is to investigate projective models of degree 2 of the supersingular  $K3$  surface  $X$  with Artin invariant 1 in characteristic 5. It is well-known that the Fermat sextic double plane in characteristic 5 is isomorphic to  $X$ . Starting from this projective model, we obtain many other projective models of degree 2, and describe birational morphisms between them. Each of these projective models exhibits an interesting geometry with peculiarity in characteristic 5.

Our method is purely computational, and can be easily adapted to the study of any  $K3$  surface in arbitrary characteristic, provided that a set of generators of the Néron-Severi lattice is explicitly given.

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We fix terminologies and explain our motivation. Let  $Y$  be a  $K3$  surface defined over an algebraically closed field of arbitrary characteristic. Let  $(\ , \ )_{\text{NS}}$  denote the intersection form of the Néron-Severi lattice  $\text{NS}(Y)$  of  $Y$ . Note that  $\text{NS}(Y)$  is identified with the Picard group of  $Y$ . For a vector  $v \in \text{NS}(Y)$ , we denote by  $\mathcal{L}_v \rightarrow Y$  the corresponding line bundle. Let  $d$  be an even positive integer. We say that a vector  $h \in \text{NS}(Y)$  is a *polarization of degree  $d$*  if  $(h, h)_{\text{NS}}$  is equal to  $d$  and the complete linear system  $|\mathcal{L}_h|$  is non-empty and has no fixed-components. Let  $h$  be a polarization of degree  $d$ . Then  $|\mathcal{L}_h|$  is base-point free by Corollary 3.2 of [16], and hence defines a morphism  $\Phi_h$  from  $Y$  to a projective space of dimension  $1 + d/2$ . We denote by

$$Y \xrightarrow{\phi_h} Y_h \xrightarrow{\psi_h} \mathbb{P}^{1+d/2}$$

the Stein factorization of  $\Phi_h$ . By [1, 2], the normal surface  $Y_h$  has only rational double points as its singularities, and  $\phi_h$  is a contraction of an  $ADE$ -configuration of smooth rational curves. We say that  $\psi_h : Y_h \rightarrow \mathbb{P}^{1+d/2}$  is the *projective model* of  $Y$  corresponding to the polarization  $h$ . We put

$$\mathcal{P}_d(Y) := \{ h \in \text{NS}(Y) \mid h \text{ is a polarization of degree } d \}.$$

The automorphism group  $\text{Aut}(Y)$  of  $Y$  acts on  $\mathcal{P}_d(Y)$ . For  $h, h' \in \mathcal{P}_d(Y)$ , we say that  $h$  and  $h'$  are *projectively equivalent* and write  $h \sim h'$  if there exist an isomorphism  $Y_h \xrightarrow{\sim} Y_{h'}$  and a linear automorphism  $\mathbb{P}^{1+d/2} \xrightarrow{\sim} \mathbb{P}^{1+d/2}$  that make the following diagram commutative:

$$(1.1) \quad \begin{array}{ccc} Y_h & \xrightarrow{\psi_h} & \mathbb{P}^{1+d/2} \\ \downarrow \wr & & \downarrow \wr \\ Y_{h'} & \xrightarrow{\psi_{h'}} & \mathbb{P}^{1+d/2}. \end{array}$$

It is obvious that, if  $h$  and  $h'$  are in the same  $\text{Aut}(Y)$ -orbit, then  $h \sim h'$  holds. Conversely, if  $h \sim h'$ , then the isomorphism  $Y_h \xrightarrow{\sim} Y_{h'}$  in (1.1) induces an automorphism  $\gamma$  of  $Y$  such that  $\gamma^*(h') = h$ . Hence the equivalence classes of the relation  $\sim$  are just the orbits of the action of  $\text{Aut}(Y)$  on  $\mathcal{P}_d(Y)$ . Let  $\text{Aut}(Y, h)$  denote the stabilizer subgroup of  $h$ . Then  $\text{Aut}(Y, h)$  is the projective automorphism group of the projective model  $\psi_h : Y_h \rightarrow \mathbb{P}^{1+d/2}$ . It is usually easy to determine  $\text{Aut}(Y, h)$ . Hence it is important to study the equivalence classes of the relation  $\sim$  for the study of  $\text{Aut}(Y)$ . Moreover, to obtain an element of  $\text{Aut}(Y)$  *not* contained in  $\text{Aut}(Y, h)$ , we need to write the isomorphism  $Y_h \xrightarrow{\sim} Y_{h'}$  in (1.1) explicitly.

We concentrate upon the supersingular  $K3$  surface  $X$  with Artin invariant 1 in characteristic 5. It is known that  $X$  has a projective model  $\psi_F : X_F \rightarrow \mathbb{P}^2$  of degree 2, where  $X_F$  is defined by

$$(1.2) \quad X_F := \{ w^2 = x^6 + y^6 + z^6 \} \subset \mathbb{P}(3, 1, 1, 1)$$

in the weighted projective space  $\mathbb{P}(3, 1, 1, 1)$  equipped with homogeneous coordinates  $[w : x : y : z]$  of weights 3, 1, 1, 1, and  $\psi_F$  is given by  $[w : x : y : z] \mapsto [x : y : z]$ . Then  $\psi_F$  is the double covering of  $\mathbb{P}^2$  branching along the Fermat sextic curve

$$B_F : x^6 + y^6 + z^6 = 0.$$

We denote by  $h_F \in \text{NS}(X)$  a polarization of the projective model  $\psi_F : X_F \rightarrow \mathbb{P}^2$ , and by

$$\Phi_F : X \xrightarrow{\phi_F} X_F \xrightarrow{\psi_F} \mathbb{P}^2$$

the Stein factorization of the morphism given by  $|\mathcal{L}_{h_F}|$ . Note that  $\phi_F : X \rightarrow X_F$  is an isomorphism. The projective automorphism group  $\text{Aut}(X, h_F)$  is an extension of the projective automorphism group  $\text{PGU}_3(\mathbb{F}_{25})$  of the plane curve  $B_F \subset \mathbb{P}^2$  by  $\text{Gal}(X_F/\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$ . In particular, the order of  $\text{Aut}(X, h_F)$  is 756,000. Using this projective model  $\psi_F : X_F \rightarrow \mathbb{P}^2$ , we obtain a set of generators of  $\text{NS}(X)$  (see Section 2). It turns out that  $\text{NS}(X)$  is generated by the numerical equivalence classes of curves on  $X_F$  defined over  $\mathbb{F}_{25}$ . As an easy consequence, we obtain the following:

**Proposition 1.1.** *Every projective model of  $X$  is projectively equivalent to a projective model defined over  $\mathbb{F}_{25}$ .*

Moreover, the Frobenius action of  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  on  $X_F$  induces an action of  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  on  $\text{NS}(X)$ , which we denote by  $v \mapsto \bar{v}$ . Note that, if  $h$  is a polarization, then so is  $\bar{h}$ . Note also that, if  $h \sim h'$  holds for polarizations  $h$  and  $h'$ , then  $\bar{h} \sim \bar{h}'$  holds. Hence  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  acts on the set of  $\text{Aut}(X)$ -orbits in  $\mathcal{P}_d(X)$ .

We will study the polarizations of degree 2. For  $h \in \mathcal{P}_2(X)$ , we denote by  $B_h \subset \mathbb{P}^2$  the branch curve of the double covering  $\psi_h : X_h \rightarrow \mathbb{P}^2$ . A  $(-2)$ -curve on  $X$  is a smooth rational curve on  $X$ . We say that a  $(-2)$ -curve  $C$  on  $X$  is  $h$ -exceptional if  $C$  is mapped to a point by the morphism  $\Phi_h : X \rightarrow \mathbb{P}^2$ . A  $(-2)$ -curve  $C$  on  $X$  is said to be an  $h$ -line if  $\Phi_h$  maps  $C$  to a line on  $\mathbb{P}^2$  isomorphically. A line  $l$  on  $\mathbb{P}^2$  is said to be  $h$ -splitting if  $l$  is the image of an  $h$ -line by  $\Phi_h$ . In other words, a line  $l \subset \mathbb{P}^2$  is  $h$ -splitting if and only if either  $l$  is an irreducible component of  $B_h$ , or  $l \not\subset B_h$  and the intersection multiplicity at each point of  $l \cap B_h$  is even.

We consider neighborhoods

$$\mathcal{B}_r := \{ v \in \text{NS}(X) \mid (v, h_F)_{\text{NS}} \leq r \}$$

of  $h_F$  in  $\text{NS}(X)$ . Since  $\overline{h_F} = h_F$ ,  $\text{Aut}(X, h_F)$  and  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  act on  $\mathcal{B}_r$ , and hence on  $\mathcal{P}_2(X) \cap \mathcal{B}_r$ . By computer-aided investigation of the vectors in  $\mathcal{P}_2(X) \cap \mathcal{B}_5$ , we have obtained the following results:

**Theorem 1.2.** *For each  $h \in \mathcal{P}_2(X) \cap \mathcal{B}_5$ , the lattice  $\text{NS}(X)$  is generated by the classes of  $h$ -exceptional curves and  $h$ -lines.*

**Theorem 1.3.** *The set  $\mathcal{P}_2(X) \cap \mathcal{B}_5$  consists of 146,945,851 vectors, and they are decomposed into the equivalence classes  $\mathcal{E}_0, \dots, \mathcal{E}_{64}$  under the relation  $\sim$ . The details of these equivalence classes are described in Section 8.*

We explain the items of the table in Section 8.

- $\mathcal{E}_i = \overline{\mathcal{E}}_j$  means that  $\mathcal{E}_i$  is equal to the image of  $\mathcal{E}_j$  under the action of  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  on  $\mathcal{P}_2(X) \cap \mathcal{B}_5$  defined above. In particular,  $\mathcal{E}_i = \overline{\mathcal{E}}_i$  means that  $\mathcal{E}_i$  is self-conjugate, while  $\mathcal{E}_i = \overline{\mathcal{E}}_{i+1}$  means that  $\mathcal{E}_i$  is *not* self-conjugate, that the items RT, |aut|, spl, N and stabs explained below are the same for  $\mathcal{E}_i$  and  $\mathcal{E}_{i+1}$ , and that the defining equation of  $B_h$  for  $\mathcal{E}_{i+1}$  is obtained from that for  $\mathcal{E}_i$  by changing the sign of  $\sqrt{2}$ .
- RT denotes the ADE-type of the singular points of  $B_h$ . Observe that  $B_h$  is singular except for the polarizations in  $\mathcal{E}_0$ , and that only nodes and cusps appear on  $B_h$  for any  $h \in \mathcal{P}_2(X) \cap \mathcal{B}_5$ .
- |aut| denotes the order of the projective automorphism group of the plane curve  $B_h \subset \mathbb{P}^2$ . Hence the order of  $\text{Aut}(X, h)$  is equal to  $2|\text{aut}|$ .
- spl is the numbers of  $h$ -splitting lines. It turns out that  $B_h$  does not contain a line for any  $h \in \mathcal{P}_2(X) \cap \mathcal{B}_5$ . We denote by  $s_{a,b}$  the number of  $h$ -splitting lines that pass through exactly  $a$  cusps of  $B_h$  and  $b$  nodes of  $B_h$ . Then spl is the list  $[s_{3,0}, s_{2,1}, s_{1,2}, s_{0,3}, s_{2,0}, s_{1,1}, s_{0,2}, s_{1,0}, s_{0,1}, s_{0,0}]$ .
- N is the total number of the vectors in  $\mathcal{E}_i \subset \mathcal{P}_2(X) \cap \mathcal{B}_5$ .
- stabs describe the orbit decomposition of  $\mathcal{E}_i$  by the action of  $\text{Aut}(X, h_F)$ . For example, the list  $[3, 12]_4, [1, 1, 1, 1, 1, 2, 2]_5$  in  $\mathcal{E}_1 = \overline{\mathcal{E}}_1$  means that  $\text{Aut}(X, h_F)$  decomposes  $\{h \in \mathcal{E}_1 \mid (h, h_F)_{\text{NS}} = 4\}$  into two orbits with the stabilizer subgroups of order 3 and 12, and  $\{h \in \mathcal{E}_1 \mid (h, h_F)_{\text{NS}} = 5\}$  into eight orbits with the stabilizer subgroups of order 1, 1, 1, 1, 1, 2 and 2.
- h is a sample element of  $\mathcal{E}_i$  written in a row vector with respect to the basis of  $\text{NS}(X)$  given in Section 2.
- An affine defining equation of the branch curve  $B_h$  with coefficients in  $\mathbb{F}_{25}$  is given in the framed box.

In fact, we establish a method to write the birational morphism  $\phi_h : X \rightarrow X_h$  explicitly as a list of rational functions on  $X \cong X_F$  for any  $h \in \mathcal{P}_2(X)$ . Applying this method to a polarization  $h \in \mathcal{E}_0$  with  $(h_F, h)_{\text{NS}} = 4$ , we obtain the following:

**Theorem 1.4.** *Let  $(w, x, y)$  be the affine coordinates of  $\mathbb{P}(3, 1, 1, 1)$  with  $z = 1$  in (1.2). Then the rational map  $g : X_F \rightarrow \mathbb{P}(3, 1, 1, 1)$  given by*

$$(w, x, y) \mapsto [\omega(w, x, y) : \xi_0(w, x, y) : \xi_1(w, x, y) : \xi_2(w, x, y)],$$

where  $\omega, \xi_0, \xi_1, \xi_2$  are the polynomials presented in Table 1.1, induces an automorphism of  $X_F$  with order 2 such that  $(h_F, g^*h_F)_{\text{NS}} = 4$ . In particular, this automorphism  $g$  is not contained in  $\text{Aut}(X, h_F)$ .

In this table,  $a + b\sqrt{2} \in \mathbb{F}_{25}$  with  $0 \leq a, b < 5$  is denoted by  $\bar{ab}$ .

$\omega := wf_\omega + h_\omega$ , where

$$\begin{aligned} f_\omega := & \bar{10}x^{12} + \bar{23}x^{11}y + \bar{21}x^{10}y^2 + \bar{02}x^9y^3 + \bar{11}x^8y^4 + \bar{33}x^7y^5 + \bar{22}x^6y^6 + \bar{40}x^5y^7 + \bar{14}x^4y^8 + \\ & \bar{13}x^2y^{10} + \bar{12}xy^{11} + \bar{24}y^{12} + \bar{22}x^{11} + \bar{11}x^{10}y + \bar{44}x^9y^2 + \bar{42}x^8y^3 + \bar{44}x^7y^4 + \bar{31}x^6y^5 + \bar{40}x^5y^6 + \\ & \bar{24}x^4y^7 + \bar{41}x^3y^8 + \bar{24}x^2y^9 + \bar{24}xy^{10} + \bar{33}y^{11} + \bar{03}x^{10} + \bar{10}x^9y + \bar{43}x^8y^2 + \bar{43}x^7y^3 + \bar{34}x^6y^4 + \\ & \bar{20}x^5y^5 + \bar{21}x^3y^7 + \bar{30}x^2y^8 + \bar{10}xy^9 + \bar{24}y^{10} + \bar{34}x^9 + \bar{23}x^8y + \bar{02}x^7y^2 + \bar{10}x^6y^3 + \bar{14}x^5y^4 + \\ & \bar{31}x^4y^5 + \bar{23}x^3y^6 + \bar{03}x^2y^7 + \bar{23}xy^8 + \bar{22}y^9 + \bar{20}x^8 + \bar{23}x^7y + \bar{32}x^6y^2 + \bar{44}x^5y^3 + \bar{42}x^3y^5 + \bar{12}x^2y^6 + \\ & \bar{22}xy^7 + \bar{42}y^8 + \bar{33}x^7 + \bar{12}x^6y + \bar{20}x^5y^2 + \bar{01}x^4y^3 + \bar{44}x^3y^4 + \bar{13}x^2y^5 + \bar{31}xy^6 + \bar{02}y^7 + \bar{30}x^6 + \\ & \bar{31}x^5y + \bar{33}x^4y^2 + \bar{23}x^3y^3 + \bar{31}x^2y^4 + \bar{41}xy^5 + \bar{31}y^6 + \bar{40}x^5 + \bar{32}x^4y + \bar{24}x^3y^2 + \bar{12}x^2y^3 + \bar{44}xy^4 + \\ & \bar{13}y^5 + \bar{14}x^3y + \bar{34}x^2y^2 + \bar{30}y^4 + \bar{31}x^3 + \bar{30}x^2y + \bar{41}xy^2 + \bar{43}y^3 + \bar{42}x^2 + \bar{40}xy + \bar{23}y^2 + \bar{03}x + \bar{02}y, \end{aligned}$$

and

$$\begin{aligned} h_\omega := & \bar{10}x^{15} + \bar{23}x^{14}y + \bar{21}x^{13}y^2 + \bar{02}x^{12}y^3 + \bar{11}x^{11}y^4 + \bar{33}x^{10}y^5 + \bar{02}x^9y^6 + \bar{04}x^8y^7 + \bar{22}x^7y^8 + \\ & \bar{11}x^6y^9 + \bar{24}x^5y^{10} + \bar{20}x^4y^{11} + \bar{12}x^3y^{12} + \bar{43}x^2y^{13} + \bar{13}xy^{14} + \bar{34}y^{15} + \bar{22}x^{14} + \bar{11}x^{13}y + \bar{44}x^{12}y^2 + \\ & \bar{42}x^{11}y^3 + \bar{44}x^{10}y^4 + \bar{31}x^9y^5 + \bar{01}x^8y^6 + \bar{02}x^7y^7 + \bar{22}x^6y^8 + \bar{20}x^5y^9 + \bar{03}x^4y^{10} + \bar{03}x^3y^{11} + \\ & \bar{10}x^2y^{12} + \bar{21}xy^{13} + \bar{41}y^{14} + \bar{03}x^{13} + \bar{10}x^{12}y + \bar{43}x^{11}y^2 + \bar{43}x^{10}y^3 + \bar{34}x^9y^4 + \bar{20}x^8y^5 + \bar{04}x^7y^6 + \\ & \bar{40}x^6y^7 + \bar{22}x^5y^8 + \bar{20}x^4y^9 + \bar{24}x^3y^{10} + \bar{02}x^2y^{11} + \bar{41}xy^{12} + \bar{31}y^{13} + \bar{34}x^{12} + \bar{23}x^{11}y + \bar{02}x^{10}y^2 + \\ & \bar{10}x^9y^3 + \bar{14}x^8y^4 + \bar{31}x^7y^5 + \bar{10}x^6y^6 + \bar{01}x^5y^7 + \bar{44}x^4y^8 + \bar{20}x^2y^{10} + \bar{43}xy^{11} + \bar{41}y^{12} + \bar{20}x^{11} + \\ & \bar{23}x^{10}y + \bar{32}x^9y^2 + \bar{44}x^8y^3 + \bar{01}x^6y^5 + \bar{42}x^5y^6 + \bar{22}x^4y^7 + \bar{33}x^3y^8 + \bar{43}x^2y^9 + \bar{02}xy^{10} + \bar{21}y^{11} + \\ & \bar{33}x^{10} + \bar{12}x^9y + \bar{20}x^8y^2 + \bar{01}x^7y^3 + \bar{22}x^6y^4 + \bar{40}x^5y^5 + \bar{41}x^4y^6 + \bar{23}x^3y^7 + \bar{30}x^2y^8 + \bar{20}xy^9 + \\ & \bar{04}y^{10} + \bar{10}x^9 + \bar{40}x^8y + \bar{41}x^7y^2 + \bar{24}x^6y^3 + \bar{42}x^5y^4 + \bar{33}x^4y^5 + \bar{42}x^3y^6 + \bar{02}x^2y^7 + \bar{22}xy^8 + \\ & \bar{13}y^9 + \bar{01}x^8 + \bar{10}x^7y + \bar{14}x^6y^2 + \bar{23}x^5y^3 + \bar{43}x^4y^4 + \bar{43}x^3y^5 + \bar{01}x^2y^6 + \bar{20}xy^7 + \bar{44}y^8 + \bar{04}x^7 + \\ & \bar{11}x^6y + \bar{03}x^5y^2 + \bar{12}x^4y^3 + \bar{44}x^3y^4 + \bar{30}x^2y^5 + \bar{22}xy^6 + \bar{20}y^7 + \bar{03}x^6 + \bar{22}x^4y^2 + \bar{41}x^3y^3 + \\ & \bar{22}x^2y^4 + \bar{14}xy^5 + \bar{12}y^6 + \bar{32}x^5 + \bar{11}x^4y + \bar{30}x^3y^2 + \bar{02}x^2y^3 + \bar{22}xy^4 + \bar{21}y^5 + \bar{04}x^4 + \bar{22}x^3y + \\ & \bar{10}x^2y^2 + \bar{04}xy^3 + \bar{13}y^4 + \bar{13}x^3 + \bar{32}x^2y + \bar{31}xy^2 + \bar{32}y^3 + \bar{03}x^2 + \bar{42}xy + \bar{44}y^2 + \bar{43}x + \bar{43}y. \end{aligned}$$

$\xi_0 := wf_0 + h_0$ , where

$$\begin{aligned} f_0 := & \bar{40}x^2 + \bar{14}xy + \bar{41}y^2 + \bar{11}x + \bar{13}y + \bar{31}, \text{ and} \\ h_0 := & \bar{40}x^5 + \bar{14}x^4y + \bar{41}x^3y^2 + \bar{12}xy^4 + \bar{30}y^5 + \bar{11}x^4 + \bar{13}x^3y + \bar{34}xy^3 + \bar{03}y^4 + \bar{31}x^3 + \\ & \bar{04}xy^2 + \bar{22}y^3 + \bar{12}y^2 + \bar{34}x + \bar{34}y + \bar{43}. \end{aligned}$$

$\xi_1 := wf_1 + h_1$ , where

$$\begin{aligned} f_1 := & \bar{10}xy + \bar{44}y^2 + \bar{20}y + \bar{21}, \text{ and} \\ h_1 := & \bar{10}x^4y + \bar{44}x^3y^2 + \bar{12}x^2y^3 + \bar{12}xy^4 + \bar{12}y^5 + \bar{20}x^3y + \bar{42}x^2y^2 + \bar{32}y^4 + \bar{21}x^3 + \\ & \bar{03}x^2y + \bar{02}xy^2 + \bar{33}y^3 + \bar{24}x^2 + \bar{43}xy + \bar{44}y^2 + \bar{21}x + \bar{43}y + \bar{01}. \end{aligned}$$

$\xi_2 := wf_2 + h_2$ , where

$$\begin{aligned} f_2 := & \bar{42}y^2 + \bar{10}x + \bar{40}y + \bar{01}, \text{ and} \\ h_2 := & \bar{42}x^3y^2 + \bar{04}x^2y^3 + \bar{14}xy^4 + \bar{14}y^5 + \bar{10}x^4 + \bar{40}x^3y + \bar{43}x^2y^2 + \bar{04}y^4 + \bar{01}x^3 + \\ & \bar{34}x^2y + \bar{10}xy^2 + \bar{04}y^3 + \bar{03}x^2 + \bar{41}xy + \bar{32}y^2 + \bar{33}x + \bar{02}y + \bar{02}. \end{aligned}$$

TABLE 1.1. The non-projective automorphism  $g$  of  $X_F$

Since the vectors  $h \in \mathcal{E}_0$  with  $(h_F, h)_{\text{NS}} = 4$  constitute a single  $\text{Aut}(X, h_F)$ -orbit, we obtain the following:

**Corollary 1.5.** *Let  $O_F \subset \text{NS}(X)$  denote the  $\text{Aut}(X)$ -orbit containing  $h_F$ , which is identified with the set of  $\text{Aut}(X, h_F)$ -cosets in  $\text{Aut}(X)$ , and let  $\mathcal{F}$  be the subset of  $O_F$  consisting of  $h \in O_F$  such that there exists a sequence  $h_0 := h_F, h_1, \dots, h_{N-1}, h_N := h$  of vectors in  $O_F$  with  $(h_{i-1}, h_i)_{\text{NS}} = 4$  for  $i = 1, \dots, N$ . Then the subgroup of  $\text{Aut}(X)$  generated by  $\text{Aut}(X, h_F)$  and the involution  $g$  in Theorem 1.4 is equal to the inverse image of  $\mathcal{F}$  by the natural map  $\text{Aut}(X) \rightarrow O_F$ .  $\square$*

We hope that, by extending the range of search for polarizations  $h$  with  $h \sim h_F$ , we can find a full set of generators of  $\text{Aut}(X)$ . To determine the necessary range of the search, we will use the arguments of Kondo and Dolgachev [9, 5]. We have already conducted a preliminary investigation of a larger neighborhood  $\mathcal{B}_6$ . A *non-exhaustive* computation shows that  $\mathcal{P}_2(X) \cap \mathcal{B}_6$  contains at least  $4 \times 10^9$  vectors. To deal with these polarizations, we need faster computer programs.

The study of singularities of sextic double plane models of complex  $K3$  surfaces using lattice theory and computer-aided calculation was initiated by Urabe [21] and Yang [22]. The idea of  $h$ -splitting lines was used in [20] for the classification of Zariski pairs of simple sextic curves. On the other hand, in [18] and [19], sextic double plane models of supersingular  $K3$  surfaces were studied by lattice theory. A shortcoming of the method in these works is that it gives only combinatorial data of the singularities of the projective models, and does not yield their defining equations explicitly.

The new devices in this article are the following: (i) Using the ample class  $h_F \in \text{NS}(X)$ , we can determine whether a given vector  $v \in \text{NS}(X)$  is nef or not (see Section 4). (ii) The fact that the classes of  $h_F$ -lines span  $\text{NS}(X)$  enables us to calculate the equation of  $X_h$  explicitly and algorithmically. (iii) To deal with the large number of polarizations, we decompose them into the  $\text{Aut}(X, h_F)$ -orbits and calculate the projective model only for a representative polarization of each orbit.

This paper is organized as follows. In Section 2, we give a set of  $h_F$ -lines whose classes form a basis of  $\text{NS}(X)$ . In Section 3, we present algorithms that can be applied to lattices in general. In Section 4, we apply them to  $\text{NS}(X)$  and describe algorithms to calculate geometric data of  $X$ . In Section 5, we explain how to calculate the morphisms  $\phi_h : X \rightarrow X_h$  and  $\psi_h : X_h \rightarrow \mathbb{P}^2$  for a given polarization  $h \in \mathcal{P}_2(X)$ . In Sections 6 and 7, the computation we carried out to prove Theorems 1.2, 1.3 and 1.4 are explained in detail. Section 8 is for the list of projective models. In the last section, we give examples of double plane models of  $X$  obtained by another method that has stemmed from [14].

**Notation.** (1) A *lattice* is a free  $\mathbb{Z}$ -module  $L$  of finite rank with a non-degenerate symmetric bilinear form  $(\ , \ )_L : L \times L \rightarrow \mathbb{Z}$ . We use the same notation  $(\ , \ )_L$  to denote the induced bilinear forms on  $L \otimes \mathbb{Q}$  and  $L \otimes \mathbb{R}$ .

(2) The transpose of a matrix  $M$  or a vector  $\mathbf{v}$  is denoted by  ${}^tM$  or  ${}^t\mathbf{v}$ .

(3) The numerical equivalence class of a divisor  $D$  on  $X$  is denoted by  $[D] \in \text{NS}(X)$ . The intersection number of divisors  $D$  and  $D'$  is written as  $(D, D')_{\text{NS}}$ .

## 2. THE NÉRON-SEVERI LATTICE OF $X$

Recall that  $B_F \subset \mathbb{P}^2$  is the Fermat curve of degree 6 in characteristic 5, which is the branch curve of the projective model  $\psi_F : X_F \rightarrow \mathbb{P}^2$  corresponding to the polarization  $h_F \in \text{NS}(X)$  of degree 2. We denote by  $B_F(\mathbb{F}_{25})$  the set of  $\mathbb{F}_{25}$ -rational points of  $B_F$ . It is known that  $|B_F(\mathbb{F}_{25})| = 126$ .

Let  $l$  be a line on  $\mathbb{P}^2$ . Since  $B_F$  is the *Hermitian curve* over  $\mathbb{F}_{25}$ , either one of the following holds (see [17] or Chapter 23 of [6]):

- (1)  $l$  intersects  $B_F$  at distinct 6 points.
- (2)  $l$  is tangent to  $B_F$  at a point  $[a : b : c] \notin B_F(\mathbb{F}_{25})$  with intersection multiplicity 5, and intersects  $B_F$  at the point  $[a^{25} : b^{25} : c^{25}]$  transversely.
- (3)  $l$  is tangent to  $B_F$  at an  $\mathbb{F}_{25}$ -rational point  $P := [a : b : c]$  of  $B_F$  with intersection multiplicity 6.

In the case (3), the inverse image of  $l$  by the double covering  $\Phi_F : X \rightarrow \mathbb{P}^2$  decomposes into two  $h_F$ -lines  $\ell^+(P)$  and  $\ell^-(P)$  such that

$$(\ell^+(P), \ell^-(P))_{\text{NS}} = 3.$$

In the cases (1) and (2), the line  $l$  is not  $h_F$ -splitting. Hence all  $h_F$ -lines on  $X$  are obtained as  $\ell^\pm(P)$  with  $P \in B_F(\mathbb{F}_{25})$ . In particular, the number of  $h_F$ -lines on  $X$  is 252. We put

$$P_0 := [0 : 1 : 1 + \sqrt{2}] \in B_F(\mathbb{F}_{25}) \quad \text{and} \quad \ell^+(P_0) := \{x^3 - w = 0, y + (1 - \sqrt{2})z = 0\}.$$

For  $P \in B_F(\mathbb{F}_{25}) \setminus \{P_0\}$ , we choose the sign of  $\ell^\pm(P)$  in such a way that

$$(\ell^+(P), \ell^+(P_0))_{\text{NS}} = 1 \quad (\text{and hence } (\ell^-(P), \ell^+(P_0))_{\text{NS}} = 0).$$

From among these  $h_F$ -lines, we choose the 22 curves  $\ell_1, \dots, \ell_{22}$  in Table 2.1. Then their intersection matrix  $M_{\text{NS}}$  is calculated as in Table 2.2. Since  $\det M_{\text{NS}} = -25$ , the classes of  $\ell_1, \dots, \ell_{22}$  form a  $\mathbb{Z}$ -basis of  $\text{NS}(X)$ . We fix this basis throughout the paper. Each element of  $\text{NS}(X)$  is written as a *row* vector with respect to this basis. In particular, the orthogonal group  $\text{O}(\text{NS}(X))$  of the lattice  $\text{NS}(X)$  acts on  $\text{NS}(X)$  from right. Since  $h_F = [\ell^+(P)] + [\ell^-(P)]$  for any  $P \in B_F(\mathbb{F}_{25})$ , we have

$$(2.1) \quad h_F = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].$$

We calculate the vector representations of the classes of all  $h_F$ -lines.

$\ell_1 := \ell^+([0 : 1 : 1 + \sqrt{2}])$	$\ell_2 := \ell^-([0 : 1 : 1 + \sqrt{2}])$
$\ell_3 := \ell^+([0 : 1 : 1 + 4\sqrt{2}])$	$\ell_4 := \ell^+([0 : 1 : 2])$
$\ell_5 := \ell^+([0 : 1 : 3])$	$\ell_6 := \ell^+([0 : 1 : 4 + \sqrt{2}])$
$\ell_7 := \ell^+([1 : 0 : 1 + \sqrt{2}])$	$\ell_8 := \ell^+([1 : 0 : 1 + 4\sqrt{2}])$
$\ell_9 := \ell^+([1 : 0 : 2])$	$\ell_{10} := \ell^+([1 : 0 : 4 + \sqrt{2}])$
$\ell_{11} := \ell^+([1 : \sqrt{2} : 1])$	$\ell_{12} := \ell^-([1 : \sqrt{2} : 2 + 2\sqrt{2}])$
$\ell_{13} := \ell^-([1 : \sqrt{2} : 2 + 3\sqrt{2}])$	$\ell_{14} := \ell^+([1 : \sqrt{2} : 3 + 2\sqrt{2}])$
$\ell_{15} := \ell^-([1 : \sqrt{2} : 3 + 3\sqrt{2}])$	$\ell_{16} := \ell^+([1 : 2\sqrt{2} : 2\sqrt{2}])$
$\ell_{17} := \ell^+([1 : 2\sqrt{2} : 3\sqrt{2}])$	$\ell_{18} := \ell^-([1 : 2\sqrt{2} : 2 + \sqrt{2}])$
$\ell_{19} := \ell^+([1 : 2\sqrt{2} : 2 + 4\sqrt{2}])$	$\ell_{20} := \ell^+([1 : 2\sqrt{2} : 3 + \sqrt{2}])$
$\ell_{21} := \ell^+([1 : 1 + \sqrt{2} : 0])$	$\ell_{22} := \ell^+([1 : 1 + 3\sqrt{2} : 1])$

TABLE 2.1. The basis of  $\text{NS}(X)$ 

-2	3	1	1	1	1	1	1	1	1	1	1	0	0	1	0	1	1	0	1	1	1	1
3	-2	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0
1	0	-2	1	1	1	1	1	1	1	0	1	0	0	0	1	1	1	0	0	1	0	0
1	0	1	-2	1	1	0	0	0	0	1	0	1	0	1	1	1	0	0	0	0	0	0
1	0	1	1	-2	1	1	1	1	1	0	0	1	0	1	0	0	1	1	1	1	0	0
1	0	1	1	1	-2	0	0	0	0	0	1	1	1	0	0	0	0	1	1	0	1	0
1	0	1	0	1	0	-2	1	0	0	1	0	1	1	0	0	1	0	1	0	0	1	0
1	0	1	0	1	0	1	-2	0	0	1	1	0	0	1	1	0	1	0	0	0	1	0
1	0	1	0	1	0	0	0	-2	1	0	1	1	1	1	0	0	1	1	0	0	0	0
1	0	1	0	1	0	0	0	1	-2	1	1	0	0	1	0	1	0	0	0	0	1	0
1	0	0	1	0	0	1	1	0	1	-2	1	1	0	0	0	0	0	0	1	0	0	0
0	1	1	0	0	1	0	1	1	1	1	-2	1	0	0	0	1	0	1	0	0	0	0
0	1	0	1	1	1	1	0	1	0	1	1	-2	0	0	1	0	1	0	0	0	1	0
1	0	0	0	0	1	1	0	1	0	0	0	0	-2	1	0	1	0	0	0	0	0	0
0	1	0	1	1	0	0	1	1	1	0	0	0	1	-2	1	0	0	0	1	0	1	0
1	0	1	1	0	0	0	1	0	0	0	0	1	0	1	-2	0	0	1	0	0	0	0
1	0	1	1	0	0	1	0	0	1	0	1	0	1	0	0	-2	1	0	1	0	0	0
0	1	1	0	1	0	0	1	1	0	0	0	1	0	0	0	1	-2	0	1	0	1	0
1	0	0	0	1	1	1	0	1	0	0	1	0	0	0	1	0	0	-2	0	0	0	0
1	0	0	0	1	1	0	0	0	0	1	0	0	0	1	0	1	1	0	-2	0	0	0
1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	1	0
1	0	0	0	0	1	1	1	0	1	0	0	1	0	1	0	0	1	0	0	1	-2	0

TABLE 2.2. The matrix  $M_{\text{NS}}$



**Example 2.1.** The class of the  $h_F$ -line  $\ell^-([1 : 4 + 4\sqrt{2} : 0])$  is

$$[-4, -6, 3, 1, 1, 2, 1, -1, 2, 1, 1, 4, 1, 0, -3, 0, 2, -1, 3, -1, -2, -3].$$

From the action of  $\mathrm{PGU}_3(\mathbb{F}_{25})$  on the set  $B_F(\mathbb{F}_{25})$ , we can calculate the action of  $\mathrm{Aut}(X, h_F)$  on the set of  $h_F$ -lines. Using this permutation representation, we can write explicitly the linear representation

$$(2.2) \quad \mathrm{Aut}(X, h_F) \rightarrow \{ T \in \mathrm{GL}_{22}(\mathbb{Z}) \mid TM_{\mathrm{NS}} {}^t T = M_{\mathrm{NS}} \} \cong \mathrm{O}(\mathrm{NS}(X)).$$

This representation is faithful.

*Remark 2.2.* The representation (2.2) is stored in the computer by the following method. We put indices to the  $h_F$ -lines as  $\ell_1, \dots, \ell_{22}, \ell_{23}, \dots, \ell_{252}$  once and for all. Then each  $\gamma \in \mathrm{Aut}(X, h_F)$  is labelled by a list of 22 integers  $\mathbf{n}_\gamma = [n_\gamma(1), \dots, n_\gamma(22)]$  in such a way that

$$[\ell_1^\gamma, \dots, \ell_{22}^\gamma] = [\ell_{n_\gamma(1)}, \dots, \ell_{n_\gamma(22)}]$$

holds, where  $\ell_i^\gamma$  is the image of  $\ell_i$  by  $\gamma$ . Then the action of  $\gamma$  on  $\mathrm{NS}(X)$  is given by  $v \mapsto vT_\gamma$ , where  $T_\gamma$  is the  $22 \times 22$  matrix whose  $i$ -th row vector is  $[\ell_{n_\gamma(i)}]$ . Thus it is enough to put the row vectors  $[\ell_1], \dots, [\ell_{252}]$  and the 756,000 lists  $\mathbf{n}_\gamma$  in the memory. We can test the data by checking the equality  $T_\gamma M_{\mathrm{NS}} {}^t T_\gamma = M_{\mathrm{NS}}$  for each  $\gamma \in \mathrm{Aut}(X, h_F)$ .

The Galois group  $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$  also acts on the set of  $h_F$ -lines by the Frobenius action on  $X_F$ . The matrix  $\Gamma_{\mathrm{NS}}$  in Table 2.3 represents this Frobenius conjugate action  $v \mapsto \bar{v} = v\Gamma_{\mathrm{NS}}$  on  $\mathrm{NS}(X)$ .

In Remark 2.2, we have indexed the  $h_F$ -lines. For an  $h_F$ -line  $\ell_j$ , let  $j'$  denote the index of the  $h_F$ -line  $\ell_{j'}$  that is the image of  $\ell_j$  by the deck-transformation of the double covering  $\Phi_F : X \rightarrow \mathbb{P}^2$ . We calculate the permutation  $j \mapsto j'$ , and store it in the computer memory.

### 3. ALGORITHMS FOR LATTICES

**3.1. An algorithm for a positive quadratic triple.** By a *quadratic triple* of  $n$ -variables, we mean a triple  $[Q, L, c]$ , where  $Q$  is an  $n \times n$  symmetric matrix with entries in  $\mathbb{Q}$ ,  $L$  is a column vector of length  $n$  with entries in  $\mathbb{Q}$ , and  $c$  is a rational number. An element of  $\mathbb{R}^n$  is written as a row vector  $\mathbf{x} = [x_1, \dots, x_n]$ . Let  $q_T = [Q, L, c]$  be a quadratic triple. The *inhomogeneous quadratic function*

$$q_T : \mathbb{Q}^n \rightarrow \mathbb{Q}$$

associated with  $q_T$  is defined by

$$q_T(\mathbf{x}) := \mathbf{x} Q {}^t \mathbf{x} + 2 \mathbf{x} L + c.$$

0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	3	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	3	-1	0	-1	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
4	5	-2	-1	-1	-1	-1	-1	0	0	0	-1	-1	1	1	-1	0	0	-1	1	0	0
-1	-1	1	1	1	0	0	1	0	0	0	0	1	-1	0	1	-1	0	0	-1	0	0
0	-1	1	0	0	1	1	0	0	0	0	1	0	0	-1	0	0	-1	1	-1	0	0
2	2	-1	1	0	0	0	0	-1	-1	0	-1	0	0	1	1	-1	-1	0	0	0	0
2	2	0	-1	-1	-1	0	0	0	1	0	0	-1	1	0	-1	1	0	-1	0	0	0
2	3	-1	0	0	0	-1	0	0	-1	-1	-1	0	0	1	1	-1	0	0	0	0	0
1	1	0	-1	0	0	0	-1	1	1	-1	0	-1	1	0	-1	1	0	0	0	0	0
3	4	-1	0	-1	-1	-1	0	-1	0	0	-1	-1	0	1	0	0	0	-1	1	0	0
3	3	-1	0	-1	0	0	-1	-1	0	0	-1	-1	1	0	-1	1	0	-1	1	0	0
-1	-2	1	1	1	1	1	1	0	-1	0	0	1	-1	-1	1	-1	0	1	-1	0	0
-2	-3	2	1	0	2	1	-1	2	1	1	4	1	0	-3	0	1	-2	3	-2	-2	-3
3	4	-2	0	0	-1	-1	0	-1	-1	-1	-3	0	0	2	0	-1	1	-2	1	1	2

TABLE 2.3. The matrix  $\Gamma_{\text{NS}}$ 

We say that a quadratic triple  $QT = [Q, L, c]$  and its inhomogeneous quadratic function  $q_{QT}$  are *positive* or *negative* according to whether the symmetric matrix  $Q$  is positive-definite or negative-definite, respectively.

Let  $QT = [Q, L, c]$  be a *positive* quadratic triple of  $n$ -variables with  $n > 1$ . In this section, we present an algorithm to calculate the finite set

$$E(QT) := \{ \mathbf{x} \in \mathbb{Z}^n \mid q_{QT}(\mathbf{x}) \leq 0 \}.$$

Suppose that  $QT = [Q, L, c]$  is written as follows:

$$Q = \left[ \begin{array}{c|c} Q' & \mathbf{p}' \\ \hline {}^t\mathbf{p}' & r' \end{array} \right] = \left[ \begin{array}{c|c} r'' & {}^t\mathbf{p}'' \\ \hline \mathbf{p}'' & Q'' \end{array} \right], \quad L = \left[ \begin{array}{c} L' \\ \hline m' \end{array} \right] = \left[ \begin{array}{c} m'' \\ \hline L'' \end{array} \right],$$

where  $Q'$  and  $Q''$  are square matrices of size  $n-1$ ,  $\mathbf{p}'$ ,  $\mathbf{p}''$ ,  $L'$  and  $L''$  are column vectors of length  $n-1$ , and  $r'$ ,  $r''$ ,  $m'$  and  $m''$  are rational numbers. Note that, since  $Q$  is positive-definite, we have  $r' > 0$  and  $r'' > 0$ . We define a positive quadratic triple  $\text{pr}(QT)$  of  $(n-1)$ -variables by

$$\text{pr}(QT) := \left[ Q' - \frac{1}{r'}(\mathbf{p}'^t \mathbf{p}'), L' - \frac{m'}{r'} \mathbf{p}', c - \frac{m'^2}{r'} \right].$$

Then, for each  $t \in \mathbb{R}$ , the compact subset  $\{\mathbf{x} \in \mathbb{R}^n \mid q_{QT}(\mathbf{x}) \leq t\}$  of  $\mathbb{R}^n$  is mapped by the projection  $[x_1, \dots, x_n] \mapsto [x_1, \dots, x_{n-1}]$  to the compact subset

$$\{\mathbf{y} \in \mathbb{R}^{n-1} \mid q_{\text{pr}(QT)}(\mathbf{y}) \leq t\}$$

of  $\mathbb{R}^{n-1}$ . For  $\mathbf{a} \in \mathbb{Q}$ , we define a positive quadratic triple  $\iota^*(\mathbf{a}, QT)$  of  $(n-1)$ -variables by

$$\iota^*(\mathbf{a}, QT) := [Q'', a\mathbf{p}'' + L'', a^2 r'' + 2am'' + c],$$

and, for  $\mathbf{a} = [a_1, \dots, a_m] \in \mathbb{Q}^m$  with  $m < n$ , we define a positive quadratic triple  $\iota^*(\mathbf{a}, QT)$  of  $(n-m)$ -variables by

$$QT^0 := QT, \quad QT^{\nu+1} := \iota^*(a_{\nu+1}, QT^\nu) \quad (\nu = 0, \dots, m-1), \quad \iota^*(\mathbf{a}, QT) := QT^m.$$

Then the positive inhomogeneous quadratic function

$$q_{\iota^*(\mathbf{a}, QT)} : \mathbb{Q}^{n-m} \rightarrow \mathbb{Q}$$

is equal to the composite  $q_{QT} \circ \iota_{\mathbf{a}}$ , where  $\iota_{\mathbf{a}}$  is the inclusion  $\mathbb{Q}^{n-m} \hookrightarrow \mathbb{Q}^n$  given by

$$[y_1, \dots, y_{n-m}] \mapsto [a_1, \dots, a_m, y_1, \dots, y_{n-m}].$$

Suppose that an element  $\mathbf{a} = [a_1, \dots, a_{n-1}]$  of  $E(\text{pr}(QT))$  is given. The positive quadratic triple  $\iota^*(\mathbf{a}, QT)$  is of *one* variable, and hence we can regard elements of  $E(\iota^*(\mathbf{a}, QT))$  as integers. The fiber of the projection  $E(QT) \rightarrow E(\text{pr}(QT))$  over  $\mathbf{a}$  is equal to

$$\{[a_1, \dots, a_{n-1}, b] \mid b \in E(\iota^*(\mathbf{a}, QT))\}.$$

Since the set  $E(\iota^*(\mathbf{a}, QT))$  is easily calculated, we can obtain  $E(QT)$  if we know  $E(\text{pr}(QT))$ . Using this idea iteratively, we carry out the following computation.

Starting from the given positive quadratic triple  $QT_n^0 := QT$  of  $n$ -variables, we compute positive quadratic triples  $QT_\mu^0$  of  $\mu$ -variables by

$$QT_\mu^0 := \text{pr}(QT_{\mu+1}^0) \quad (\mu = n-1, \dots, 1).$$

We prepare an empty set  $E := \{\}$ . We then write a program  $\mathcal{Q}(\nu, \mathbf{a})$  that takes an integer  $\nu \leq n+1$  and a vector  $\mathbf{a} = [a_1, \dots, a_{\nu-1}] \in \mathbb{Z}^{\nu-1}$  as input, and carries out the task below. Note that, when  $\mathcal{Q}(\nu, \mathbf{a})$  starts with  $\nu > 1$ ,  $\mathbf{a}$  is an element of  $E(QT_{\nu-1}^0)$ , and for  $\mu > \nu-1$ ,  $QT_\mu^{\nu-1}$  is the positive quadratic triple  $\iota^*(\mathbf{a}, QT_\mu^0)$  of  $(\mu - \nu + 1)$ -variables. In particular,  $QT_\nu^{\nu-1}$  is of one variable.

The task of  $\mathcal{Q}(\nu, \mathbf{a})$ :

- (1) If  $\nu = n + 1$ , then  $\mathcal{Q}(\nu, \mathbf{a})$  appends  $\mathbf{a}$  to the set  $E$ .
- (2) If  $\nu \leq n$ , then the program  $\mathcal{Q}(\nu, \mathbf{a})$ 
  - (2-i) calculates the set  $E(QT_\nu^{\nu-1}) = \{b_1, \dots, b_N\}$ , and
  - (2-ii) for each  $b_i \in E(QT_\nu^{\nu-1})$ ,
    - (2-ii-a) computes  $QT_\mu^\nu := \iota^*(b_i, QT_\mu^{\nu-1})$  for  $\mu = \nu + 1, \dots, n$ , and
    - (2-ii-b) proceeds to execute  $\mathcal{Q}(\nu + 1, [a_1, \dots, a_{\nu-1}, b_i])$ .

We execute  $\mathcal{Q}(1, [\ ])$ . Since each  $E(QT_\nu^{\nu-1})$  is finite, this program certainly terminates. When the whole computation halts, the set  $E$  is equal to  $E(QT)$ .

**3.2. An application to hyperbolic lattices I.** Changing the sign, we can apply the algorithm of the previous subsection to *negative* inhomogeneous quadratic functions. Since the Néron-Severi lattices of smooth algebraic surfaces are hyperbolic, we will use the algorithm in this way.

Let  $N$  be a lattice of rank  $n$ . Suppose that  $N$  is hyperbolic, that is, the signature of  $(\ , \ )_N$  is  $(1, n - 1)$ . Let  $\{[v_i, a_i] \mid i = 1, \dots, k\}$  be a finite set of pairs of  $v_i \in N$  and  $a_i \in \mathbb{Z}$  such that

$$(3.1) \quad (v_i, v_i)_N > 0 \text{ for at least one } i,$$

and let  $d$  be an integer. We can calculate the set

$$(3.2) \quad \{x \in N \mid (x, v_i)_N = a_i \text{ for } i = 1, \dots, k, \text{ and } (x, x)_N = d\}$$

by the following method. We put

$$\begin{aligned} M_0 &:= \{x \in N \mid (x, v_i)_N = 0 \text{ for } i = 1, \dots, k\}, \\ M &:= \{x \in N \mid (x, v_i)_N = a_i \text{ for } i = 1, \dots, k\}. \end{aligned}$$

It is easy to obtain a basis  $b_1, \dots, b_r$  of the  $\mathbb{Z}$ -submodule  $M_0$  of  $N$ . It is also easy to determine whether  $M$  is empty or not, and, in the case  $M \neq \emptyset$ , we can find an element  $c \in M$ . Suppose that  $M \neq \emptyset$ . Then we identify  $M$  with  $\mathbb{Z}^r$  by the affine-linear isomorphism

$$(x_1, \dots, x_r) \mapsto c + x_1 b_1 + \dots + x_r b_r$$

from  $\mathbb{Z}^r$  to  $M$ . By the assumption (3.1), the restriction of  $(\ , \ )_N$  to  $M \subset N$  defines a negative inhomogeneous quadratic function  $q_{\mathbb{Z}^r} : \mathbb{Z}^r \rightarrow \mathbb{Z}$ . Therefore we can calculate the set  $\{x \in \mathbb{Z}^r \mid q_{\mathbb{Z}^r}(x) = d\}$  by the algorithm in Section 3.1, and hence the set (3.2) is computed.

**3.3. An application to hyperbolic lattices II.** Let  $N$  be the hyperbolic lattice in the previous subsection. Suppose that we are given vectors  $h, v \in N$  satisfying

$$(3.3) \quad (h, h)_N > 0, \quad (v, v)_N > 0, \quad (h, v)_N > 0.$$

We describe an algorithm that calculates, for a given integer  $d$ , the set

$$(3.4) \quad S := \{r \in N \mid (r, h)_N > 0, (r, v)_N < 0, (r, r)_N = d\}.$$

Consider the orthogonal direct-sum decomposition

$$N \otimes \mathbb{R} = \langle h \rangle \oplus \langle h \rangle^\perp.$$

We denote the second projection by

$$\text{pr}_2 : N \otimes \mathbb{R} \rightarrow \langle h \rangle^\perp,$$

and put

$$W := \text{pr}_2(N),$$

which is a free  $\mathbb{Z}$ -module of rank  $n - 1$  such that  $W \otimes \mathbb{R} = \langle h \rangle^\perp$ . Note that  $W \subset N \otimes \mathbb{Q}$ . We denote by

$$(\ , \ )_W : W \times W \rightarrow \mathbb{Q}$$

the restriction of  $(\ , \ )_N$  to  $W$ . Suppose that  $x \in N \otimes \mathbb{R}$  satisfies  $(h, x)_N \neq 0$  and  $(x, x)_N > 0$ . Then the composite

$$(3.5) \quad \langle x \rangle^\perp \hookrightarrow N \otimes \mathbb{R} \xrightarrow{\text{pr}_2} \langle h \rangle^\perp$$

is an isomorphism of  $\mathbb{R}$ -vector spaces. Let

$$\varphi_x : \langle h \rangle^\perp \xrightarrow{\sim} \langle x \rangle^\perp$$

denote the inverse of the isomorphism (3.5), that is,

$$\varphi_x(y) = y - \frac{(y, x)_N}{(h, x)_N} h \quad \text{for } y \in \langle h \rangle^\perp.$$

We then define  $f_x : \langle h \rangle^\perp \rightarrow \mathbb{R}$  by

$$f_x(y) := (\varphi_x(y), \varphi_x(y))_N = (y, y)_W + \frac{(y, x)_N^2}{(h, x)_N^2} (h, h)_N \quad \text{for } y \in \langle h \rangle^\perp = W \otimes \mathbb{R}.$$

Note that, since  $(x, x)_N > 0$ , the real quadratic form  $(\ , \ )_N$  restricted to  $\langle x \rangle^\perp$  is negative-definite, and hence so is  $f_x$ . By the condition (3.3), we see that  $f_{h+tv}$  is negative-definite on  $W \otimes \mathbb{R}$  for any  $t \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ . (Here we understand that  $f_{h+\infty v} = f_v$ .)

For simplicity, we put

$$c_h := (h, h)_N, \quad c_v := (h, v)_N, \quad v_W := \text{pr}_2(v) \in W.$$

Let  $x'$  be a vector in  $\langle h \rangle^\perp = W \otimes \mathbb{R}$ . Since  $v - v_W \in \langle h \rangle$ , we have

$$(3.6) \quad f_{h+tv}(x') = (x', x')_W + \frac{t^2(x', v_W)_W^2}{(c_h + tc_v)^2} c_h.$$

By (3.3), we have  $c_h/c_v > 0$ , and hence, for a fixed  $x' \in \langle h \rangle^\perp$ ,  $f_{h+tv}(x')$  is a non-decreasing function with respect to  $t \in \mathbb{R}_{\geq 0}$  bounded from above by

$$f_{h+\infty v}(x') = (x', x')_W + \frac{(x', v_W)_W^2}{c_v^2} c_h.$$

Note that  $f_{h+\infty v}$  restricted to  $W \subset W \otimes \mathbb{R}$  is  $\mathbb{Q}$ -valued, and hence  $f_{h+\infty v}$  is a negative inhomogeneous quadratic function on  $W \otimes \mathbb{Q}$ . Applying the algorithm in Section 3.1 to  $f_{h+\infty v}$ , we can calculate the finite set

$$S_W := \{ r' \in W \mid f_{h+\infty v}(r') \geq d \},$$

where  $d$  is the integer given as input.

Suppose that  $r$  is an element of the set  $S$  in (3.4). We put

$$t_r := -\frac{(r, h)_N}{(r, v)_N} \in \mathbb{R}_{>0}.$$

Then we have  $r \in \langle h + t_r v \rangle^\perp$ . We put  $r' := \text{pr}_2(r) \in W$ . Since  $\varphi_{h+t_r v}(r') = r$ , we have

$$d = (r, r)_N = f_{h+t_r v}(r') \leq f_{h+\infty v}(r').$$

Therefore  $r' \in S_W$  holds. Let  $\rho \in \mathbb{Q}$  be the rational number such that  $r = \rho h + r'$ . Since  $(r, r)_N = d$ ,  $(r', h)_N = 0$  and  $(r, h)_N > 0$ , we have

$$(3.7) \quad \rho = \frac{(r, h)_N}{c_h} = \sqrt{\frac{d - (r', r')_W}{c_h}}.$$

The right-hand side of (3.7) can be calculated if we know  $r' \in W$ .

Therefore we obtain  $S$  from  $S_W$  by the following method. First we set  $S = \{ \}$ . For each  $r' \in S_W$ , we put

$$\rho' := \sqrt{\frac{d - (r', r')_W}{c_h}} \quad \text{and} \quad r := \rho' h + r' \in N \otimes \mathbb{R}.$$

We then determine whether  $r$  is contained in  $N$  or not. (If  $\rho' \notin \mathbb{Q}$ , then we obviously have  $r \notin N$ .) If  $r \in N$ ,  $(r, h)_N > 0$  and  $(r, v)_N < 0$ , we append  $r$  to  $S$ . When this task is done for all  $r' \in S_W$ , the set  $S$  is equal to the set (3.4).

#### 4. GEOMETRIC APPLICATIONS

We apply the algorithms in the previous section to the hyperbolic lattice  $\text{NS}(X)$ . The existence of the ample class  $h_F \in \text{NS}(X)$  provides us with finite procedures to calculate various geometric data of projective models of  $X$ .

**4.1. Polarizations.** If  $v \in \text{NS}(X)$  is a polarization, then we necessarily have  $(v, v)_{\text{NS}} > 0$  and  $(v, h_F)_{\text{NS}} > 0$ .

**Proposition 4.1.** *Suppose that a vector  $v \in \text{NS}(X)$  satisfies  $(v, v)_{\text{NS}} > 0$  and  $(v, h_F)_{\text{NS}} > 0$ . Then  $v$  is a polarization if and only if the sets*

$$\begin{aligned} S_1 &:= \{ r \in \text{NS}(X) \mid (r, r)_{\text{NS}} = -2, (r, h_F)_{\text{NS}} > 0, (r, v)_{\text{NS}} < 0 \} \quad \text{and} \\ S_2 &:= \{ e \in \text{NS}(X) \mid (e, e)_{\text{NS}} = 0, (e, v)_{\text{NS}} = 1 \} \end{aligned}$$

*are both empty.*

*Proof.* First note that  $(v, v)_{\text{NS}} > 0$  and  $(v, h_F)_{\text{NS}} > 0$  imply that  $v$  is effective.

The condition  $S_1 = \emptyset$  is equivalent to the condition that  $v$  be nef. Indeed, suppose that  $v$  is nef. If  $(r, r)_{\text{NS}} = -2$  and  $(r, h_F)_{\text{NS}} > 0$ , then  $r$  is effective and hence  $(r, v)_{\text{NS}} \geq 0$ . Therefore  $S_1 = \emptyset$ . Conversely, suppose that  $v$  is *not* nef. Then there exists an irreducible curve  $C$  such that  $(v, C)_{\text{NS}} < 0$ . Since  $v$  is effective, we have  $|C| = \{C\}$ , and therefore  $(C, C)_{\text{NS}} < 0$ . Thus  $C$  is a  $(-2)$ -curve and  $[C] \in S_1$ .

Suppose that  $v$  is a polarization of degree  $d$ . Then  $v$  is nef and hence  $S_1 = \emptyset$ . If  $e \in S_2$ , then  $(e, e)_{\text{NS}} = 0$  and  $(e, v)_{\text{NS}} > 0$  imply that  $e$  is the class of a divisor  $E$  such that  $\dim |E| > 0$ . Let  $M$  be a general member of the moving part  $|M|$  of  $|E|$ . Then  $(e, v)_{\text{NS}} = 1$  implies that  $\Phi_v : X \rightarrow \mathbb{P}^{1+d/2}$  maps  $M$  to a line isomorphically. Hence  $M$  is a  $(-2)$ -curve, which is a contradiction. Therefore  $S_2 = \emptyset$ .

Conversely, suppose that  $v$  is not a polarization and  $S_1 = \emptyset$ . Since  $v$  is nef with  $(v, v)_{\text{NS}} > 0$ , we see from Proposition 0.1 of [11] that the complete linear system  $|\mathcal{L}_v|$  is written as  $m|E| + \Gamma$ , where  $m = 1 + (v, v)_{\text{NS}}/2$ ,  $|E|$  is an elliptic pencil, and  $\Gamma$  is a  $(-2)$ -curve such that  $(E, \Gamma)_{\text{NS}} = 1$ . Then we have  $[E] \in S_2$ .  $\square$

The sets  $S_1$  and  $S_2$  can be calculated by the algorithms in Sections 3.3 and 3.2, respectively. Hence Proposition 4.1 enables us to determine whether a given vector  $v \in \text{NS}(X)$  is a polarization or not.

**4.2.  $h$ -Exceptional curves.** Let  $h \in \text{NS}(X)$  be a polarization of arbitrary degree. A  $(-2)$ -curve  $C$  on  $X$  is called  *$h$ -exceptional* if  $\Phi_h$  contracts  $C$ . The set  $\text{Exc}(h) \subset \text{NS}(X)$  of the classes of  $h$ -exceptional curves is calculated by the following algorithm. We calculate the finite set

$$R := \{r \in \text{NS}(X) \mid (r, r)_{\text{NS}} = -2, (r, h)_{\text{NS}} = 0\}$$

by the algorithm in Section 3.2, and classify the elements of  $R$  by the degree with respect to the ample class  $h_F$  as follows:

$$R[m] := \{r \in R \mid (r, h_F)_{\text{NS}} = m\} \quad \text{and} \quad R^+ := \bigcup_{m>0} R[m].$$

We say that  $r \in R^+$  is *indecomposable* if there are no vectors  $r_1, \dots, r_k \in R^+$  with  $k > 1$  such that  $r = r_1 + \dots + r_k$ . Since each  $R[m]$  is finite, we can determine whether a given vector  $r \in R^+$  is indecomposable or not by comparing  $r$  with all vectors of the form  $r_1 + \dots + r_k$ , where  $r_1 \in R[m_1], \dots, r_k \in R[m_k]$  and  $m_1 + \dots + m_k = (r, h_F)_{\text{NS}}$ . It is obvious that  $\text{Exc}(h) \subset R^+$ . If  $r \in R^+$ , then  $r$  is effective and every reduced irreducible component of the divisor  $D$  with  $r = [D]$  is  $h$ -exceptional. Therefore  $r \in R^+$  is contained in  $\text{Exc}(h)$  if and only if  $r$  is an indecomposable element of  $R^+$ .

**4.3.  $h$ -Lines.** Let  $h \in \text{NS}(X)$  be a polarization of arbitrary degree. A  $(-2)$ -curve  $C$  on  $X$  is called an  *$h$ -line* if  $\Phi_h$  maps  $C$  to a line isomorphically. The set  $\text{Lin}(h) \subset$

$\text{NS}(X)$  of the classes of  $h$ -lines is calculated by the following algorithm. We calculate the finite set

$$L := \{ r \in \text{NS}(X) \mid (r, r)_{\text{NS}} = -2, (r, h)_{\text{NS}} = 1 \},$$

and put

$$L[m] := \{ r \in L \mid (r, h_F)_{\text{NS}} = m \}, \quad L^+ := \bigcup_{m>0} L[m].$$

It is obvious that  $\text{Lin}(h) \subset L^+$ . If  $r \in L^+$ , then we see that  $r$  is the class of an effective divisor  $D$ , that exactly one irreducible component  $D_0$  of  $D$  is an  $h$ -line, and that  $D - D_0$  is a finite sum of  $h$ -exceptional curves. Hence  $r \in L^+$  is contained in  $\text{Lin}(h)$  if and only if there are no  $r' \in L[m']$  with  $m' < (r, h_F)_{\text{NS}}$  and  $r_1, \dots, r_k \in \text{Exc}(h)$  with  $k \geq 1$  such that  $r = r' + r_1 + \dots + r_k$ . Since each of  $L[m']$  and  $\text{Exc}(h)$  are finite, we can determine the subset  $\text{Lin}(h) \subset L^+$ .

**4.4.  $h_F$ -Conics.** We say that a  $(-2)$ -curve  $C$  on  $X$  is an  $h_F$ -conic if  $\Phi_F : X \rightarrow \mathbb{P}^2$  maps  $C$  to a smooth conic isomorphically. The set  $\text{Con}(h_F) \subset \text{NS}(X)$  of the classes of  $h_F$ -conics are calculated as follows. We calculate the finite set

$$T := \{ r \in \text{NS}(X) \mid (r, r)_{\text{NS}} = -2, (r, h_F)_{\text{NS}} = 2 \}.$$

It turns out that  $|T| = 22,050$ . It is obvious that  $\text{Con}(h_F) \subset T$ . Suppose that  $r \in T$ , and let  $D$  be an effective divisor such that  $r = [D]$ . Since  $B_F$  is smooth, either one of the following holds:

- (a)  $D$  is an  $h_F$ -conic, or
- (b)  $D$  is a union of  $h_F$ -lines  $\ell$  and  $\ell'$  that intersect at one point.

We have calculated  $\text{Lin}(h_F)$  in Section 2. Therefore we can calculate

$$\text{Con}(h_F) = T \setminus \{ [\ell] + [\ell'] \mid [\ell], [\ell'] \in \text{Lin}(h_F), (\ell, \ell')_{\text{NS}} = 1 \}.$$

It turns out that  $|\text{Con}(h_F)| = 6,300$ . By means of the representation (2.2), we see that  $\text{Aut}(X, h_F)$  acts on  $\text{Con}(h_F)$  transitively.

**Definition 4.2.** We say that a smooth conic  $\Gamma \subset \mathbb{P}^2$  is *totally tangent to  $B_F$*  if  $\Gamma$  is tangent to  $B_F$  at distinct 6 points.

Using a result of B. Segre (n. 81 of [17]), which holds for any Hermitian curves, we see that there exist exactly 3,150 conics totally tangent to  $B_F$ . If  $\Gamma \subset \mathbb{P}^2$  is a conic totally tangent to  $B_F$ , then  $\Phi_F^{-1}(\Gamma)$  is a union of two  $h_F$ -conics  $\tilde{\Gamma}^+$  and  $\tilde{\Gamma}^-$ . Combining these results, we obtain the following:

**Proposition 4.3.** *The set  $\text{Con}(h_F)$  consists of  $[\tilde{\Gamma}^\pm]$ , where  $\Gamma$  are conics totally tangent to  $B_F$ .  $\square$*

**Example 4.4.** The image  $\tilde{\Gamma}_0^+$  of the map  $\mathbb{P}^1 \hookrightarrow X_F \subset \mathbb{P}(3, 1, 1, 1)$  given by

$$t \mapsto [w : x : y : z] = [2(t^5 - t) : 2\sqrt{2}t : 4\sqrt{2} + (4 + \sqrt{2})t^2 : 2 + 4\sqrt{2} + t^2]$$



is an  $h_F$ -conic. Calculating the intersection number of  $\tilde{\Gamma}_0^+$  with the  $h_F$ -lines  $\ell_1, \dots, \ell_{22}$ , we see that

$$[\tilde{\Gamma}_0^+] = [-1, -2, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, -1, 0, 1, -1, 0, 1, 0, 0, 0].$$

Moving  $\tilde{\Gamma}_0^+$  and  $[\tilde{\Gamma}_0^+]$  by  $\text{Aut}(X, h_F)$ , we obtain the parametric representations of all  $h_F$ -conics  $\tilde{\Gamma}^\pm$  and their classes  $[\tilde{\Gamma}^\pm]$ .

*Remark 4.5.* We can calculate the irreducible decomposition of  $B_h$  from  $\text{NS}(X)$  and  $h$  by the algorithm given in [22] and [20]. We did not carry out this computation, because the irreducible decomposition can be obtained from the explicit defining equation of  $B_h$  computed by the method in the next section.

## 5. EXPLICIT DEFINING EQUATIONS

We identify  $X$  with  $X_F$  by the isomorphism  $\phi_F : X \xrightarrow{\sim} X_F$ , so that, for a polarization  $h$  of degree 2, we consider  $\Phi_h : X \rightarrow \mathbb{P}^2$  and  $\phi_h : X \rightarrow X_h$  as morphisms from  $X_F$ . In this section, we explain a method to write the morphisms  $\Phi_h$  and  $\phi_h$  as lists of rational functions on  $X_F$  over  $\mathbb{F}_{25}$ , and to find the defining equation

$$w^2 = s_h(x, y, z)$$

of the projective model  $\psi_h : X_h \rightarrow \mathbb{P}^2$ .

**5.1. The global sections of a line bundle.** Let  $H_\infty \subset X_F$  denote the hyperplane section defined by  $z = 0$  in (1.2). We use the affine coordinates  $(w, x, y)$  of  $\mathbb{P}(3, 1, 1, 1)$  with  $z = 1$ , and put

$$F := w^2 - x^6 - y^6 - 1 \in \mathbb{F}_{25}[w, x, y].$$

For any  $g \in \mathbb{F}_{25}[w, x, y]$ , there exists a unique polynomial  $\bar{g}^F$  of the form  $wf + h$  with  $f, h \in \mathbb{F}_{25}[x, y]$  such that

$$g \equiv \bar{g}^F \pmod{(F)} \quad \text{in } \mathbb{F}_{25}[w, x, y].$$

We call  $\bar{g}^F$  the *normal form* of  $g$ , which is obtained from  $g$  by replacing  $w^2$  with  $x^6 + y^6 + 1$ . Let  $m$  be an integer. By identifying the line bundle  $\mathcal{L}_{mh_F} \rightarrow X$  with the invertible sheaf  $\mathcal{O}_{X_F}(mH_\infty)$ , the vector space  $\Gamma(X, \mathcal{L}_{mh_F})$  of the global sections of  $\mathcal{L}_{mh_F}$  defined over  $\mathbb{F}_{25}$  is naturally identified with the vector subspace

$$V_m := \{ wf + h \mid f, h \in \mathbb{F}_{25}[x, y], \deg f \leq m - 3, \deg h \leq m \}$$

of  $\mathbb{F}_{25}[w, x, y]$  consisting of the normal forms of weighted degree  $\leq m$ . Recall that all  $h_F$ -lines are defined over  $\mathbb{F}_{25}$ , and that no  $h_F$ -lines are contained in  $H_\infty$ . We have indexed the  $h_F$ -lines as  $\ell_1, \dots, \ell_{252}$  in Remark 2.2. For  $j = 1, \dots, 252$ , we denote by

$$I_j \subset \mathbb{F}_{25}[w, x, y]$$

the inhomogeneous ideal defining  $\ell_j$  in  $\mathbb{P}(3, 1, 1, 1)$ , and put

$$I_j^{(\nu)} := I_j^\nu + (F) \subset \mathbb{F}_{25}[w, x, y] \quad \text{for } \nu \in \mathbb{Z}_{>0}.$$

Recall from Section 2 that  $j'$  is the index of the  $h_F$ -line  $\ell_{j'}$  that is the image of  $\ell_j$  by the deck-transformation of  $X_F$  over  $\mathbb{P}^2$ .

We explain an algorithm that takes a vector  $v \in \text{NS}(X)$  as input, and calculates the vector space  $\Gamma(X, \mathcal{L}_v)$  of the global sections of the corresponding line bundle  $\mathcal{L}_v \rightarrow X$  defined over  $\mathbb{F}_{25}$ . Using the  $\mathbb{Z}$ -basis  $[\ell_1], \dots, [\ell_{22}]$  of  $\text{NS}(X)$ ,  $v$  is uniquely written as

$$v = \sum_{i \in J^+} a_i [\ell_i] - \sum_{j \in J^-} b_j [\ell_j],$$

where  $J^+$  and  $J^-$  are disjoint subsets of  $\{1, \dots, 22\}$ , and  $a_i, b_j$  are positive integers. Since  $[\ell_i] + [\ell_{i'}] = h_F$  for any  $i$ , we have

$$v = d'(v)h_F - \sum_{i \in J^+} a_i [\ell_{i'}] - \sum_{j \in J^-} b_j [\ell_j], \quad \text{where } d'(v) := \sum_{i \in J^+} a_i.$$

Thus we have an expression

$$(5.1) \quad v = d(v)h_F - \sum_{j \in J} c_j [\ell_j],$$

where  $d(v)$  is a non-negative integer,  $J$  is a subset of  $\{1, \dots, 252\}$ , and  $c_j$  are positive integers. Then the vector space  $\Gamma(X, \mathcal{L}_v)$  is identified with the space of global sections of  $\mathcal{O}_{X_F}(d(v)H_\infty)$  that vanish along  $\ell_j$  with order  $c_j$  for each  $j \in J$ , that is,

$$(5.2) \quad \Gamma(X, \mathcal{L}_v) \cong V_{d(v)} \cap \bigcap_{j \in J} I_j^{(c_j)},$$

where the intersections are taken in  $\mathbb{F}_{25}[w, x, y]$ . From now on, we regard  $\Gamma(X, \mathcal{L}_v)$  as a subspace of  $V_{d(v)}$  by (5.2).

*Remark 5.1.* The integer  $d(v)$  and the set  $J$  in (5.1) are *not* uniquely determined by  $v$ , because of the linear relations among  $[\ell_j]$ .

The vector space  $V_{d(v)}$  has a basis

$$m_\alpha := wM \text{ or } N \quad (\alpha = 1, \dots, 2 + d(v)^2),$$

where  $M$  and  $N$  are the monomials of  $x$  and  $y$  with  $\deg M \leq d(v) - 3$  and  $\deg N \leq d(v)$ . We calculate the Gröbner basis  $G_j$  of the ideal  $I_j^{(c_j)} \subset \mathbb{F}_{25}[w, x, y]$  for each  $j \in J$ . (In the actual calculation, we used the graded reverse lexicographic order **grevlex**( $w, x, y$ ). See p. 56 of [4].) We then calculate the remainders  $\overline{m_\alpha}^{G_j}$  of the monomials  $m_\alpha$  by these Gröbner bases  $G_j$ . An element  $\sum_\alpha u_\alpha m_\alpha$  of  $V_{d(v)}$  with  $u_\alpha \in \mathbb{F}_{25}$  is contained in  $\Gamma(X, \mathcal{L}_v)$  if and only if

$$\sum_\alpha u_\alpha \overline{m_\alpha}^{G_j} = 0 \quad \text{for each } j \in J.$$

These equalities constitute a system of linear equations with unknowns  $u_\alpha$ . Solving these equations, we obtain a basis of  $\Gamma(X, \mathcal{L}_v)$  as a list of polynomials in  $V_{d(v)}$ .

Let  $k$  be a positive integer. Then we can write the vector  $kv \in \text{NS}(X)$  as

$$kv := kd(v)h_F - \sum_{j \in J} kc_j[\ell_j]$$

using the same  $d(v)$  and  $J$  that appeared in (5.1). Under this choice, the natural homomorphism

$$\Gamma(X, \mathcal{L}_v)^{\otimes k} \rightarrow \Gamma(X, \mathcal{L}_{kv})$$

is given by restricting the linear homomorphism

$$g_1 \otimes \cdots \otimes g_k \mapsto \overline{g_1 \cdots g_k}^F$$

from  $V_{d(v)}^{\otimes k}$  to  $V_{kd(v)}$ .

**5.2. The morphisms  $\Phi_h$  and  $\phi_h$ .** We explain an algorithm that takes a vector  $h \in \mathcal{P}_2(X)$  as input, and calculates the morphisms  $\Phi_h$ ,  $\phi_h$  and a defining equation of  $X_h$  in  $\mathbb{P}(3, 1, 1, 1)$ .

Let  $h \in \text{NS}(X)$  be a polarization of degree 2. Then we have

$$\dim \Gamma(X, \mathcal{L}_h) = 3, \quad \dim \Gamma(X, \mathcal{L}_{3h}) = 11, \quad \dim \Gamma(X, \mathcal{L}_{6h}) = 38.$$

We find an expression  $h = d(h)h_F - \sum_{j \in J} c_j[\ell_j]$  of  $h$  in the form (5.1). By the method described above, we obtain three polynomials

$$\xi_i(w, x, y) \in V_{d(h)} \quad (i = 0, 1, 2)$$

that form a basis of  $\Gamma(X, \mathcal{L}_h)$ . The rational map  $(w, x, y) \mapsto [\xi_0 : \xi_1 : \xi_2]$  gives the morphism  $\Phi_h : X_F \rightarrow \mathbb{P}^2$ .

Next we calculate eleven polynomials that form a basis of  $\Gamma(X, \mathcal{L}_{3h}) \subset V_{3d(h)}$  using the expression  $3h = 3d(h)h_F - \sum_{j \in J} 3c_j[\ell_j]$ . We compute the normal forms

$$\overline{\xi_i \xi_{i'} \xi_{i''}}^F \quad (i, i', i'' \in \{0, 1, 2\})$$

of the ten polynomials  $\xi_i \xi_{i'} \xi_{i''}$ . These normal forms are contained in  $\Gamma(X, \mathcal{L}_{3h})$ . Then we find a polynomial  $\omega \in V_{3d(h)}$  that is contained in  $\Gamma(X, \mathcal{L}_{3h})$ , but is *not* contained in the 10-dimensional subspace spanned by  $\overline{\xi_i \xi_{i'} \xi_{i''}}^F$ . The rational map

$$(w, x, y) \mapsto [\omega : \xi_0 : \xi_1 : \xi_2] \in \mathbb{P}(3, 1, 1, 1)$$

gives the morphism  $\phi_h : X_F \rightarrow X_h$ .

We then compute the 39 normal forms

$$\overline{\omega^2}^F, \quad \overline{\omega \xi_i \xi_{i'} \xi_{i''}}^F, \quad \overline{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_6}}^F \quad (i, i', i'', i_1, \dots, i_6 \in \{0, 1, 2\}),$$

which are contained in  $\Gamma(X, \mathcal{L}_{6h}) \subset V_{6d(h)}$ . Since  $\dim \Gamma(X, \mathcal{L}_{6h}) = 38$ , there exists a non-trivial linear relation over  $\mathbb{F}_{25}$  among these 39 polynomials. Using homogeneous polynomials  $b(x, y, z)$  of degree 3 and  $c(x, y, z)$  of degree 6 with coefficients in  $\mathbb{F}_{25}$ , we write this linear relation as

$$(5.3) \quad \overline{a \omega^2 + b(\xi_0, \xi_1, \xi_2) \omega + c(\xi_0, \xi_1, \xi_2)}^F = 0,$$

where  $a \in \mathbb{F}_{25}$ . If  $a = 0$ , then the rational function  $\omega$  on  $X_F$  would be invariant under the deck-transformation of  $X_F$  over  $\mathbb{P}^2$ , and hence  $\omega$  would be written as a cubic polynomial of  $\xi_0, \xi_1, \xi_2$  in the integral domain  $\mathbb{F}_{25}[w, x, y]/(F)$ . Therefore we may assume that  $a = 1$ . We then replace  $\omega$  by

$$\omega - 2 \overline{b(\xi_0, \xi_1, \xi_2)}^F \in V_{3d(h)},$$

and calculate the homogeneous polynomial  $s_h(x, y, z)$  of degree 6 with coefficients in  $\mathbb{F}_{25}$  by

$$s_h(x, y, z) := -b(x, y, z)^2 - c(x, y, z),$$

so that the linear relation (5.3) is now written as

$$\overline{\omega^2}^F = \overline{s_h(\xi_0, \xi_1, \xi_2)}^F.$$

The projective model  $\psi_h : X_h \rightarrow \mathbb{P}^2$  is defined by  $w^2 = s_h(x, y, z)$ , and the branch curve  $B_h \subset \mathbb{P}^2$  is defined by  $s_h(x, y, z) = 0$ .

*Remark 5.2.* The computational difficulty of this method grows rapidly as  $d(h)$  increases.

**5.3. The projective equivalence.** Let  $\overline{\mathbb{F}}$  denote the algebraic closure of  $\mathbb{F}_{25}$ . For  $T \in \mathrm{GL}_3(\overline{\mathbb{F}})$ , we denote by  $[T] \in \mathrm{PGL}_3(\overline{\mathbb{F}})$  the image of  $T$  by the natural homomorphism  $\mathrm{GL}_3(\overline{\mathbb{F}}) \rightarrow \mathrm{PGL}_3(\overline{\mathbb{F}})$ , and by  $P \mapsto P^{[T]}$  the linear transformation of  $\mathbb{P}^2$  given by  $[a : b : c] \mapsto [a : b : c]T$ . Let  $\mathbb{H}_6$  denote the set of homogeneous polynomials of degree 6 in variables  $x, y, z$  with coefficients in  $\mathbb{F}_{25}$ . For  $f \in \mathbb{H}_6 \otimes \overline{\mathbb{F}}$ , we put

$$f^T(x, y, z) := f(x', y', z'), \quad \text{where } (x', y', z') = (x, y, z)T^{-1}.$$

If  $f = 0$  defines a curve  $C \subset \mathbb{P}^2$ , then  $f^T = 0$  defines the image  $C^{[T]}$  of the curve  $C$  by the projective linear transformation  $P \mapsto P^{[T]}$ .

Let  $h$  and  $h'$  be elements of  $\mathcal{P}_2(X)$ . By definition, we have the following:

$$(5.4) \quad h \sim h' \iff \text{there exist } T \in \mathrm{GL}_3(\overline{\mathbb{F}}) \text{ and } c \in \overline{\mathbb{F}}^\times \text{ such that } s_{h'} = c s_h^T.$$

The polynomials  $\omega, \xi_0, \xi_1, \xi_2$  giving  $\phi_h : X_F \rightarrow X_h$  that are obtained in the previous subsection are unique up to the following transformations:

$$\begin{aligned} \omega &\mapsto \lambda \omega, \quad \text{where } \lambda \in \mathbb{F}_{25}^\times, \\ (\xi_0, \xi_1, \xi_2) &\mapsto (\xi_0, \xi_1, \xi_2)T, \quad \text{where } T \in \mathrm{GL}_3(\mathbb{F}_{25}). \end{aligned}$$

Under this transformation, the sextic polynomial  $s_h \in \mathbb{H}_6$  is changed to  $\lambda^2 s_h^T$ . Therefore we can define the following relation  $\sim_{\mathbb{F}}$  on  $\mathcal{P}_2(X)$ :

$$(5.5) \quad h \sim_{\mathbb{F}} h' \iff \text{there exist } T \in \mathrm{GL}_3(\mathbb{F}_{25}) \text{ and } \lambda \in \mathbb{F}_{25}^\times \text{ such that } s_{h'} = \lambda^2 s_h^T.$$

We investigate the relation between  $\sim$  and  $\sim_{\mathbb{F}}$ .

**Lemma 5.3.** *Suppose that there exist  $T \in \mathrm{GL}_3(\mathbb{F}_{25})$  and  $c \in \overline{\mathbb{F}}^\times$  that satisfy  $s_{h'} = c s_h^T$ . Then  $h \sim_{\mathbb{F}} h'$  holds.*

*Proof.* Let  $K$  denote the quotient field of the integral domain  $\mathbb{F}_{25}[w, x, y]/(F)$ . Then we have  $\overline{\mathbb{F}} \cap K = \mathbb{F}_{25}$ . By the assumption  $s_{h'} = c s_h^T$ , we see that  $c \in \mathbb{F}_{25}^\times$  and that there exist non-zero elements  $\omega$  and  $\omega'$  of  $K$  such that  $\omega'^2 = c\omega^2$ . Hence  $c$  is a square in  $\mathbb{F}_{25}$ .  $\square$

Let  $B_1 = \{f_1 = 0\}$  and  $B_2 = \{f_2 = 0\}$  be reduced plane curves defined by  $f_1 \in \mathbb{H}_6$  and  $f_2 \in \mathbb{H}_6$ , respectively. We consider the set

$$\mathrm{isom}(B_1, B_2) := \{ \tau \in \mathrm{PGL}_3(\overline{\mathbb{F}}) \mid B_1^\tau = B_2 \}$$

of projective isomorphisms from  $B_1$  to  $B_2$  defined over  $\overline{\mathbb{F}}$ . By definitions and Lemma 5.3, we have

$$(5.6) \quad h \sim h' \iff \mathrm{isom}(B_h, B_{h'}) \neq \emptyset,$$

$$(5.7) \quad h \sim_{\mathbb{F}} h' \iff \mathrm{isom}(B_h, B_{h'}) \cap \mathrm{PGL}_3(\mathbb{F}_{25}) \neq \emptyset.$$

**Definition 5.4.** Let  $Q = [Q_0, Q_1, Q_2, Q_3]$  and  $Q' = [Q'_0, Q'_1, Q'_2, Q'_3]$  be two ordered 4-tuples of points of  $\mathbb{P}^2$  such that no three points of  $Q$  are colinear and no three points of  $Q'$  are colinear. Then there exists a unique projective transformation  $\tau_{QQ'} \in \mathrm{PGL}_3(\overline{\mathbb{F}})$  such that

$$Q^{\tau_{QQ'}} := [Q_0^{\tau_{QQ'}}, Q_1^{\tau_{QQ'}}, Q_2^{\tau_{QQ'}}, Q_3^{\tau_{QQ'}}]$$

is equal to  $Q'$ . Let  $T_{QQ'} \in \mathrm{GL}_3(\overline{\mathbb{F}})$  denote a matrix such that  $[T_{QQ'}] = \tau_{QQ'}$ .

Let  $B$  be a reduced plane curve defined over  $\overline{\mathbb{F}}$ . We define  $\mathcal{Q}(B)$  to be the set

$$(5.8) \quad \left\{ [Q_0, Q_1, Q_2, Q_3] \mid \begin{array}{l} Q_i \in \mathrm{Sing}(B) \text{ for } i = 0, \dots, 3, \text{ and no three of} \\ Q_0, \dots, Q_3 \text{ are colinear} \end{array} \right\}.$$

Let  $R$  be an element of  $\mathcal{Q}(B_1)$ . Then the map  $\tau \mapsto R^\tau$  induces a bijection

$$(5.9) \quad \mathrm{isom}(B_1, B_2) \cong \{ Q' \in \mathcal{Q}(B_2) \mid f_2 = c f_1^{T_{RQ'}} \text{ for some } c \in \overline{\mathbb{F}}^\times \}.$$

If all points of  $Q$  and  $Q'$  are  $\mathbb{F}_{25}$ -rational, then we have  $\tau_{QQ'} \in \mathrm{PGL}_3(\mathbb{F}_{25})$ . Hence we obtain the following:

**Lemma 5.5.** *Suppose that every singular point of  $B_h$  and  $B_{h'}$  is  $\mathbb{F}_{25}$ -rational, and that  $\mathcal{Q}(B_h)$  and  $\mathcal{Q}(B_{h'})$  are non-empty. Then  $\mathrm{isom}(B_h, B_{h'})$  is contained in  $\mathrm{PGL}_3(\mathbb{F}_{25})$ .*  $\square$

The bijection (5.9) also provides us with a practical method to calculate the group  $\mathrm{aut}(B) = \mathrm{isom}(B, B)$  for a plane curve  $B$  defined over  $\mathbb{F}_{25}$  satisfying  $\mathrm{Sing}(B) \subset \mathbb{P}^2(\mathbb{F}_{25})$  and  $\mathcal{Q}(B) \neq \emptyset$ .

## 6. PROOF OF THEOREMS 1.2 AND 1.3

**6.1. Step 1.** First note that  $\mathcal{P}_2(X) \cap \mathcal{B}_3 = \{h_F\}$ . Indeed, let  $h$  be a polarization of degree 2 such that  $d := (h, h_F)_{\text{NS}} \leq 3$ , and let  $H$  be a general member of  $|\mathcal{L}_h|$ . Then the morphism  $\Phi_F : X \rightarrow \mathbb{P}^2$  associated with  $|\mathcal{L}_{h_F}|$  maps  $H$  to a plane curve of degree  $d$  birationally, or to a plane curve of degree  $d/2$  by a morphism with mapping degree 2. Since  $H$  is of genus 2, we see that  $d = 2$  and  $h = h_F$ .

**6.2. Step 2.** We calculate the sets

$$\mathcal{V}_\delta := \{v \in \text{NS}(X) \mid (v, v)_{\text{NS}} = 2, (v, h_F)_{\text{NS}} = \delta\},$$

for  $\delta = 4$  and  $5$  by the algorithm in Section 3.2. The numbers of vectors in these sets are

$$|\mathcal{V}_4| = 1,020,600, \quad |\mathcal{V}_5| = 208,059,000.$$

We put

$$\mathcal{V} := \{h_F\} \cup \mathcal{V}_4 \cup \mathcal{V}_5.$$

Then we have  $\mathcal{P}_2(X) \cap \mathcal{B}_5 = \mathcal{P}_2(X) \cap \mathcal{V}$ .

*Remark 6.1.* The computation of  $\mathcal{V}_4$  and  $\mathcal{V}_5$  was carried out by a program written in Maple. It took about 10 days. The memory size of the result written in the plain text format is about 12.5 GB.

Our goal is to calculate the subset  $\mathcal{P}_2(X) \cap \mathcal{V}$  of  $\mathcal{V}$  and decompose it into the equivalence classes under the relation  $\sim$  of the projective equivalence. Note that  $\text{Aut}(X, h_F)$  acts on  $\mathcal{V}_4$ ,  $\mathcal{V}_5$  and  $\mathcal{P}_2(X)$ , and that, if  $h$  and  $h'$  are in the same  $\text{Aut}(X, h_F)$ -orbit, then we have  $h \sim_{\mathbb{F}} h'$ , because every element of  $\text{Aut}(X, h_F)$  is defined over  $\mathbb{F}_{25}$ .

**6.3. Step 3.** We first decompose  $\mathcal{V}$  into the  $\text{Aut}(X, h_F)$ -orbits, and choose a representative vector from each orbit.

We have embedded  $\text{Aut}(X, h_F)$  in the orthogonal group  $\text{O}(\text{NS}(X))$  by (2.2). Recall that  $\text{Aut}(X, h_F)$  acts on  $\text{NS}(X)$  from right. We introduce a total order  $<$  on  $\text{NS}(X)$  as follows. Let  $\mathbf{x} = [x_1, \dots, x_{22}]$  and  $\mathbf{y} = [y_1, \dots, y_{22}]$  be vectors in  $\text{NS}(X)$ . We put

$$\mathbf{x} <_{\text{lex}} \mathbf{y} \iff \text{there exists } k \text{ such that } x_k < y_k \text{ and } x_j = y_j \text{ for } j < k,$$

and define  $\mathbf{x} < \mathbf{y}$  by

$$\mathbf{x} < \mathbf{y} \iff \sum_{i=1}^{22} |x_i| < \sum_{i=1}^{22} |y_i| \quad \text{or} \quad \left( \sum_{i=1}^{22} |x_i| = \sum_{i=1}^{22} |y_i| \quad \text{and} \quad \mathbf{x} <_{\text{lex}} \mathbf{y} \right).$$

We then put

$$\mathcal{R} := \{v \in \text{NS}(X) \mid v \leq vT \text{ for all } T \in \text{Aut}(X, h_F)\}.$$

In other words,  $\mathcal{R}$  is the set of vectors  $v$  that are minimal in the  $\text{Aut}(X, h_F)$ -orbit containing  $v$ . Therefore we can define the representative vector  $v_o$  of each  $\text{Aut}(X, h_F)$ -orbit  $o \subset \text{NS}(X)$  by

$$o \cap \mathcal{R} = \{v_o\}.$$

We calculate the list  $\mathcal{R} \cap \mathcal{V}_4$ ,  $\mathcal{R} \cap \mathcal{V}_5$ , and the order of the stabilizer subgroup  $\text{Stab}(v) \subset \text{Aut}(X, h_F)$  for each  $v \in \mathcal{R} \cap \mathcal{V}$  by the following simple algorithm, where  $\mathbf{V} = \mathcal{V}$  and  $\mathbf{R} = \mathcal{R}$ .

```

for v in V {
    minflag:=true;
    orderStab:=0;
    for T in Aut(X, h_F) while minflag {
        vT:=v*T;
        if vT<v then {minflag:=false;}
        else if vT=v then {orderStab:=orderStab+1;}
    }
    if minflag=true, then v is an element of R and
    orderStab is the order of the stabilizer subgroup of v;
    if minflag=false, then v is not an element of R.
}

```

We obtain

$$|\mathcal{R} \cap \mathcal{V}_4| = |\mathcal{V}_4 / \text{Aut}(X, h_F)| = 8 \quad \text{and} \quad |\mathcal{R} \cap \mathcal{V}_5| = |\mathcal{V}_5 / \text{Aut}(X, h_F)| = 312.$$

*Remark 6.2.* We choose this total order  $<$  on  $\text{NS}(X)$  so that we can express each  $v \in \mathcal{R} \cap \mathcal{V}$  in the form (5.1) with  $d(v)$  small. See Remarks 5.1 and 5.2.

*Remark 6.3.* We should have

$$\sum_{v \in \mathcal{R} \cap \mathcal{V}_4} \frac{1}{|\text{Stab}(v)|} = \frac{1020600}{756000}, \quad \sum_{v \in \mathcal{R} \cap \mathcal{V}_5} \frac{1}{|\text{Stab}(v)|} = \frac{208059000}{756000}.$$

These equalities can be used in a test of the obtained data.

**6.4. Step 4.** For each  $v \in \mathcal{R} \cap \mathcal{V}$ , we calculate the  $\text{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ -conjugate  $\bar{v} = v\Gamma_{\text{NS}}$  of  $v$ , where  $\Gamma_{\text{NS}}$  is the matrix given in Table 2.3, and find the representative vector  $v^\Gamma \in \mathcal{R} \cap \mathcal{V}$  of the  $\text{Aut}(X, h_F)$ -orbit containing  $\bar{v}$ .

*Remark 6.4.* The two steps above were carried out by a program written in the C language. We confirmed that all the entries  $x \in \mathbb{Z}$  of the vectors  $v$  in  $\mathcal{V}$  and the matrices  $T$  of the representation (2.2) satisfy  $|x| < 100$ , and calculated  $vT$  and  $\bar{v}T = v\Gamma_{\text{NS}}T$  using the 32-bit `int` data type of C.

6.5. **Step 5.** For each  $v \in \mathcal{R} \cap \mathcal{V}$ , we calculate the sets  $S_1$  and  $S_2$  in Proposition 4.1, and determine whether  $v$  is a polarization or not. We obtain

$$|\mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}_4| = 7 \quad \text{and} \quad |\mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}_5| = 224.$$

**Example 6.5.** The vector

$$v = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1] \in \mathcal{R} \cap \mathcal{V}_4$$

is not a polarization, because

$$r := [0, 1]$$

satisfies  $(r, r)_{\text{NS}} = -2$ ,  $(r, h_F)_{\text{NS}} = 1 > 0$  and  $(r, v)_{\text{NS}} = -1 < 0$ .

6.6. **Step 6.** For simplicity, we put

$$\mathcal{H} := \mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}.$$

By means of the algorithms in Sections 4.2 and 4.3, we calculate, for each  $h \in \mathcal{H}$ , the set  $\text{Exc}(h)$  of the classes of  $h$ -exceptional curves, and the set  $\text{Lin}(h)$  of the classes of  $h$ -lines. From  $\text{Exc}(h)$ , we determine the  $ADE$ -type  $\text{RT}(h)$  of  $\text{Sing}(B_h)$ . By this computation, it turns out that  $B_h$  has only nodes and cusps.

Let  $D_h \in \text{O}(\text{NS}(X))$  denote the matrix such that  $v \mapsto vD_h$  is the action on  $\text{NS}(X)$  of the involution of  $X$  associated with the double covering  $\psi_h : X_h \rightarrow \mathbb{P}^2$ . The eigenspace of  $D_h$  on  $\text{NS}(X) \otimes \mathbb{Q}$  with eigenvalue 1 is spanned by

- the polarization  $h$ ,
- the classes of  $h$ -exceptional curves over the nodes of  $B_h$ , and
- $[C_1] + [C_2]$ , where  $C_1 + C_2$  are the exceptional divisors over the cusps of  $B_h$ , that is,  $C_1$  and  $C_2$  are  $h$ -exceptional curves with  $(C_1, C_2)_{\text{NS}} = 1$ .

The orthogonal complement of this subspace is the eigenspace with eigenvalue  $-1$ . Hence we can calculate  $D_h$  from  $\text{Exc}(h)$ . Note that  $D_h$  acts on  $\text{Lin}(h) \subset \text{NS}(X)$ . We confirm that every orbit of  $D_h$  on  $\text{Lin}(h)$  consists of distinct two vectors. Hence  $B_h$  does not contain any  $h$ -splitting line as an irreducible component.

Using  $\text{Exc}(h)$  and  $\text{Lin}(h)$ , we make the list

$$\text{spl}(h) = [s_{3,0}, s_{2,1}, s_{1,2}, s_{0,3}, s_{2,0}, s_{1,1}, s_{0,2}, s_{1,0}, s_{0,1}, s_{0,0}]$$

of numbers  $s_{a,b}$  of  $h$ -splitting lines that pass through exactly  $a$  cusps of  $B_h$  and  $b$  nodes of  $B_h$ .

We then confirm that the union of  $\text{Exc}(h)$  and  $\text{Lin}(h)$  spans  $\text{NS}(X)$  for any  $h \in \mathcal{H}$ . Thus Theorem 1.2 is proved.

*Remark 6.6.* Recall that  $v^\Gamma$  is the representative vector of the  $\text{Aut}(X, h_F)$ -orbit containing  $\bar{v}$ . For any  $h \in \mathcal{H}$ , we should have  $h^\Gamma \in \mathcal{H}$ , and  $\text{RT}(h^\Gamma) = \text{RT}(h)$ ,  $\text{spl}(h^\Gamma) = \text{spl}(h)$ . These can be used in a test of the data.

*Remark 6.7.* The above two steps were carried out by Maple.



**6.7. Step 7.** Next we proceed to the calculation of the projective models. For each  $h \in \mathcal{H}$ , we carry out the computation described in Section 5, and calculate polynomials  $\omega, \xi_0, \xi_1, \xi_2 \in \mathbb{F}_{25}[w, x, y]$  that give the morphism  $\phi_h : X_F \rightarrow X_h$ , and a homogeneous polynomial  $s_h(x, y, z)$  of degree 6 with coefficients in  $\mathbb{F}_{25}$  such that  $w^2 = s_h(x, y, z)$  defines  $X_h$ . Then we compute the coordinates of the singular points of  $B_h$ .

*Remark 6.8.* It turns out that, for every  $h \in \mathcal{H}$ , we have

$$\text{Exc}(h) \subset \text{Lin}(h_F) \cup \text{Con}(h_F),$$

where  $\text{Con}(h_F)$  is the set of the classes of  $h_F$ -conics on  $X$  calculated in Section 4.4. Since we have the parametric presentations  $\rho : \mathbb{P}^1 \hookrightarrow X_F$  of all  $h_F$ -lines and  $h_F$ -conics, we can calculate the coordinates of the singular points of  $B_h$  by composing  $\rho$  with  $\Phi_h : X_F \rightarrow \mathbb{P}^2$  given by the polynomials  $\xi_0, \xi_1, \xi_2$ .

*Remark 6.9.* By this computation, we observe the following fact. For any  $h \in \mathcal{H}$  with  $\text{RT}(h) \neq 0$ , every singular point of  $B_h = \{s_h = 0\}$  is  $\mathbb{F}_{25}$ -rational, and the set  $\mathcal{Q}(B_h)$  defined by (5.8) is non-empty. By Lemma 5.5, it follows that  $\text{isom}(B_h, B_{h'})$  is contained in  $\text{PGL}_3(\mathbb{F}_{25})$  for any  $h, h' \in \mathcal{H}$  with  $\text{RT}(h) \neq 0$  and  $\text{RT}(h') \neq 0$ .

*Remark 6.10.* The calculation of the Gröbner bases  $G_j$  of the ideals  $I_j^{(c_j)} \subset \mathbb{F}_{25}[w, x, y]$  was carried out by Maple. The other polynomial calculation and the linear algebra in  $\mathbb{F}_{25}[w, x, y]$  were carried out by a program written in  $\mathbb{C}$ . The computation of the coordinates of the singular points of  $B_h$  was done by Maple.

**6.8. Step 8.** We decompose  $\mathcal{H} = \mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}$  into the equivalence classes under the relation  $\sim_{\mathbb{F}}$  defined by (5.5), and confirm that the relations  $\sim$  and  $\sim_{\mathbb{F}}$  are the same on  $\mathcal{H}$ .

**6.8.1. The case where  $B_h$  is non-singular.** In  $\mathcal{H}$ , there are exactly three polarizations  $h$  such that  $\text{RT}(h) = 0$ :  $h_F$  and

$$h'_F = [1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0] \in \mathcal{V}_4, \quad \text{and}$$

$$h''_F = [0, -1, 0, 2, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, -1, 0, 0, 0, 0] \in \mathcal{V}_5.$$

We have already calculated the defining equations  $w^2 = s_{h'_F}$  and  $w^2 = s_{h''_F}$  of the projective models  $X_{h'_F}$  and  $X_{h''_F}$ , respectively. Applying the following result to the case  $n = 3$  and  $q = 5$ , we see that  $h'_F \sim_{\mathbb{F}} h_F$  and  $h''_F \sim_{\mathbb{F}} h_F$ .

**Theorem 6.11** (n. 3 of [17]). *Let  $q$  be a power of a prime integer  $p$ . Then the image of the map  $\text{GL}_n(\mathbb{F}_{q^2}) \rightarrow \text{GL}_n(\mathbb{F}_{q^2})$  given by  $M \mapsto M {}^t \overline{M}$  is equal to the set*

$$\{ H \in \text{GL}_n(\mathbb{F}_{q^2}) \mid H = {}^t \overline{H} \}$$

*of non-degenerate Hermitian matrices over  $\mathbb{F}_{q^2}$ , where  $\overline{H}$  is obtained from  $H$  by applying  $a \mapsto a^q$  to the entries.*

**Corollary 6.12.** *For  $h \in \mathcal{P}_2(X)$ , we have  $h \sim_{\mathbb{F}} h_F$  if and only if there exist a  $3 \times 3$  non-degenerate Hermitian matrix  $(a_{ij})$  over  $\mathbb{F}_{25}$  and  $\lambda \in \mathbb{F}_{25}^\times$  such that  $s_h = s_h(x_0, x_1, x_2)$  is of the form  $\lambda^2 \sum_{i,j=0}^2 a_{ij} x_i x_j^5$ .*

6.8.2. *The case where  $B_h$  is singular.* We introduce a total order  $\prec$  on the set  $\mathbb{H}_6$  of homogeneous sextic polynomials of variables  $x, y, z$  with coefficients in  $\mathbb{F}_{25}$ .

*Remark 6.13.* Any total order will do. We have chosen the following one. For  $\alpha = a + \sqrt{2}b \in \mathbb{F}_{25}$  with  $0 \leq a, b < 5$ , we put  $|\alpha| := a + 5b \in \mathbb{Z}$ . For

$$f = \sum_{i+j+k=6} \alpha_{ijk} x^i y^j z^k \in \mathbb{H}_6,$$

we denote by  $\text{nz}(f)$  the number of non-zero monomials in  $f$ , and put

$$c(f) := [|\alpha_{600}|, |\alpha_{510}|, |\alpha_{501}|, |\alpha_{420}|, |\alpha_{411}|, \dots, |\alpha_{024}|, |\alpha_{015}|, |\alpha_{006}|] \in \mathbb{Z}^{28},$$

where the coefficients  $\alpha_{ijk}$  in  $c(f)$  are sorted according to the monomial order  $\text{grevlex}(z, y, x)$ . Then we put  $f \prec g$  if and only if

$$\text{nz}(f) < \text{nz}(g) \text{ or } \left( \begin{array}{l} \text{nz}(f) = \text{nz}(g) \text{ and the leftmost non-zero component of } \\ c(f) - c(g) \text{ is } < 0 \end{array} \right).$$

We fix four reference points

$$P_0 := [1 : 0 : 0], \quad P_1 := [0 : 1 : 0], \quad P_2 := [0 : 0 : 1], \quad P_3 := [1 : 1 : 1],$$

and put  $P := [P_0, P_1, P_2, P_3]$ . Recall that  $\mathcal{Q}(B_h)$  is defined by (5.8) and  $\tau_{QP} \in \text{PGL}_3(\overline{\mathbb{F}})$  is defined in Definition 5.4. For  $h \in \mathcal{H}$  with  $\text{RT}(h) \neq 0$ , we put

$$\mathcal{T}(h) := \{ \tau \in \text{PGL}_3(\overline{\mathbb{F}}) \mid \text{Sing}(B_h^\tau) \ni P_i \text{ for } i = 0, 1, 2, 3 \} = \{ \tau_{QP} \mid Q \in \mathcal{Q}(B_h) \},$$

$$S(h) := \{ \lambda^2 s_h^T \mid \lambda \in \mathbb{F}_{25}^\times, \quad T \in \text{GL}_3(\mathbb{F}_{25}) \},$$

$$S^P(h) := \{ s'_h \in S(h) \mid \text{the curve } s'_h = 0 \text{ is singular at } P_0, \dots, P_3 \}.$$

By Remark 6.9, we have  $\mathcal{T}(h) \subset \text{PGL}_3(\mathbb{F}_{25})$  and  $\mathcal{T}(h) \neq \emptyset$ , and hence

$$S^P(h) = \{ \lambda^2 s_h^T \mid \lambda \in \mathbb{F}_{25}^\times, \quad T \in \text{GL}_3(\mathbb{F}_{25}), \quad [T] \in \mathcal{T}(h) \} \neq \emptyset$$

holds. Since  $\mathcal{Q}(B_h)$  is easily calculated, so is  $S^P(h)$ . (Note that, since  $|\text{GL}_3(\mathbb{F}_{25})|$  is very large, it is difficult to calculate  $S(h)$ .) We put

$$s_h^{\min} := \text{the minimal element of } S^P(h) \text{ with respect to } \prec.$$

By definition, we have  $h \sim_{\mathbb{F}} h'$  if and only if  $S(h) = S(h')$ . Hence we have

$$h \sim_{\mathbb{F}} h' \iff s_h^{\min} = s_{h'}^{\min}.$$

By this method, we decompose  $\mathcal{H}$  into the equivalence classes of the relation  $\sim_{\mathbb{F}}$ .

Remark 6.9 combined with (5.6), (5.7) and the observation in Section 6.8.1 imply that the two relations  $\sim$  and  $\sim_{\mathbb{F}}$  define the same relation on  $\mathcal{H}$ . Thus the equivalence classes  $\mathcal{E}_0, \dots, \mathcal{E}_{64}$  of  $\sim$  are obtained.

For  $h \in \mathcal{H}$ , we denote by  $[h] \subset \mathcal{H}$  the equivalence class of  $\sim$  containing  $h$ , and by  $s_{[h]}$  the polynomial  $s_h^{\min}$  obtained above. We also denote by  $B_{[h]}$  the plane curve defined by  $s_{[h]} = 0$ .

*Remark 6.14.* Since the calculations in Step 8 do not use the results of Step 6, we can use the fact that  $h \sim h'$  implies  $\text{RT}(h) = \text{RT}(h')$  and  $\text{spl}(h) = \text{spl}(h')$  in the test of the data.

**6.9. Step 9.** For each equivalence class  $[h] \subset \mathcal{H}$ , we calculate the group  $\text{aut}(B_{[h]}) = \text{isom}(B_{[h]}, B_{[h]})$  and the set  $\text{isom}(B_{[h]}, \overline{B_{[h]}})$  by the method given in Section 5.3, where  $\overline{B_{[h]}}$  is the plane curve defined by the polynomial  $\overline{s_{[h]}} \in \mathbb{H}_6$  obtained from  $s_{[h]}$  by  $\sqrt{2} \mapsto -\sqrt{2}$ .

*Remark 6.15.* In the test of the data, we can use the fact that  $[h] = [h^\Gamma]$  holds if and only if  $\text{isom}(B_{[h]}, \overline{B_{[h]}}) \neq \emptyset$ .

**6.10. Step 10.** We search for  $(T, \lambda) \in \text{GL}_3(\mathbb{F}_{25}) \times \mathbb{F}_{25}^\times$  such that  $\lambda^2 s_{[h]}^T$  has coefficients in  $\mathbb{F}_5$ . If such  $(T, \lambda)$  exists, then we necessarily have  $h \sim h^\Gamma$ .

**Proposition 6.16.** *For  $f \in \mathbb{H}_6$ , the following conditions are equivalent.*

- (i) There exist  $T \in \text{GL}_3(\mathbb{F}_{25})$  and  $\lambda \in \mathbb{F}_{25}^\times$  such that  $\lambda^2 f^T$  has coefficients in  $\mathbb{F}_5$ .
- (ii) There exist  $M \in \text{GL}_3(\mathbb{F}_{25})$  and  $c \in \mathbb{F}_{25}^\times$  such that  $f^M = c \bar{f}$ ,  $M \overline{M} = \text{Id}_3$  and  $c^3 = 1$ .

Since we have already calculated the set  $\text{isom}(B_{[h]}, \overline{B_{[h]}})$  for every  $[h] \subset \mathcal{H}$ , we can make the list of  $(M, c) \in \text{GL}_3(\mathbb{F}_{25}) \times \mathbb{F}_{25}^\times$  such that  $s_{[h]}^M = c \overline{s_{[h]}}$ . Therefore we can determine whether the condition (ii) is satisfied or not for  $f = s_{[h]}$ . The proof below shows how to find  $(T, \lambda)$  in the condition (i) from  $(M, c)$  in the condition (ii).

*Proof of Proposition 6.16.* Suppose that (i) holds. Since  $\bar{\lambda}^2 \bar{f}^T = \lambda^2 f^T$ , we have  $(\lambda^{-1} \bar{\lambda})^2 \bar{f} = f^{T \overline{T}^{-1}}$ . Then  $M := T \overline{T}^{-1}$  and  $c := (\lambda^{-1} \bar{\lambda})^2 = \lambda^8$  satisfy the equalities in (ii). Conversely, suppose that (ii) holds. Then there exists  $T \in \text{GL}_3(\mathbb{F}_{25})$  such that  $M = T \overline{T}^{-1}$ . Indeed, let  $m : \mathbb{F}_{25}^3 \rightarrow \mathbb{F}_{25}^3$  be the map given by

$$m(\mathbf{x}) := \mathbf{x} + \bar{\mathbf{x}} M,$$

where vectors of  $\mathbb{F}_{25}^3$  are written as row vectors. Then there exist  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  such that  $m(\mathbf{x}_1), m(\mathbf{x}_2), m(\mathbf{x}_3)$  are linearly independent. Let  $C$  denote the  $3 \times 3$  matrix whose row vectors are  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ . We put

$$S := C + \overline{C} M,$$

which is non-degenerate. Then we have  $\overline{S} = S M^{-1}$ . Therefore, putting  $T := \overline{S}^{-1}$ , we have  $M = T \overline{T}^{-1}$ . Since  $f^M = c \bar{f}$ , we have  $f^T = c \overline{f^T}$ . Since  $c^3 = 1$ , there exists  $\lambda \in \mathbb{F}_{25}^\times$  such that  $c = \lambda^8 = (\lambda^{-1} \bar{\lambda})^2$ . Then we have  $\lambda^2 f^T = \bar{\lambda}^2 \overline{f^T}$ , and hence  $\lambda^2 f^T$  has coefficients in  $\mathbb{F}_5$ .  $\square$

**Example 6.17.** Consider the equivalence class  $\mathcal{E}_7 = \overline{\mathcal{E}}_7$ . The branch curve  $B_{[h]}$  is defined by

$$\sqrt{2}x^3y^3 + (1 + 3\sqrt{2})x^2y^4 + x^4 + (2 + 2\sqrt{2})x^3y + (1 + 4\sqrt{2})x^2y^2 + xy^3 + (2 + 2\sqrt{2})y^4 + \sqrt{2}x^2 + (1 + 3\sqrt{2})xy = 0,$$

and it has six nodes at  $[0 : 0 : 1]$ ,  $[0 : 1 : 0]$ ,  $[1 : 1 : 1]$ ,  $[1 : 1 : 4]$ ,  $[1 : 2 + 2\sqrt{2} : 2]$ ,  $[1 : 2 + 2\sqrt{2} : 3]$  and a cusp at  $[1 : 0 : 0]$ . The set  $\text{isom}(B_{[h]}, \overline{B_{[h]}})$  consists of the two projective transformations  $[M_1]$  and  $[M_2]$  given by the matrices

$$M_1 := \begin{bmatrix} 4 + 4\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 + 2\sqrt{2} \end{bmatrix} \quad \text{and} \quad M_2 := \begin{bmatrix} 1 + \sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 + 2\sqrt{2} \end{bmatrix}.$$

Since neither  $M_1\overline{M}_1$  nor  $M_2\overline{M}_2$  is a scalar matrix, there exist no polynomials with coefficients in  $\mathbb{F}_5$  that defines a curve projectively isomorphic to  $B_{[h]}$ .

*Remark 6.18.* Except for the equivalence class  $\mathcal{E}_7 = \overline{\mathcal{E}}_7$ , we have found a defining equation  $s_{\mathbb{F},[h]}$  of  $B_h$  with coefficients in  $\mathbb{F}_5$  for each  $\mathcal{E}_n$  with  $\mathcal{E}_n = \overline{\mathcal{E}}_n$ .

*Remark 6.19.* Steps 8, 9 and 10 were carried out by Maple.

#### 6.11. Remarks.

*Remark 6.20.* The defining equations of  $B_h$  given in Section 8 are chosen as follows.

*The case where there exists a defining equation  $s_{\mathbb{F},[h]}$  with coefficients in  $\mathbb{F}_5$ .* We choose the minimal element of the set

$$\{ c s_{\mathbb{F},[h]}^T \mid c \in \mathbb{F}_5^\times, T \in \text{GL}_3(\mathbb{F}_5) \}$$

with respect to the total order  $\prec$  in Remark 6.13. This equation is *not* uniquely determined by  $h$ , because there are pairs of curves  $B$  and  $B'$  defined over  $\mathbb{F}_5$  that are projectively isomorphic over  $\mathbb{F}_{25}$  but are not projectively isomorphic over  $\mathbb{F}_5$ .

*Other cases.* We put

$$\begin{aligned} P^{(0)} &:= \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]\}, \\ P^{(1)} &:= \{[0 : 1 : 1], [0 : 1 : -1], [1 : 0 : 1], [1 : 0 : -1]\}, \\ P^{(2)} &:= \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 2]\}. \end{aligned}$$

We choose the minimal element of the set

$$\{ \lambda^2 s_{[h]}^T \mid \lambda \in \mathbb{F}_{25}^\times, T \in \text{GL}_3(\mathbb{F}_{25}), \text{Sing}(B_{[h]}^{[T]}) \supset P^{(i)} \text{ for some } i \}$$

with respect to the total order  $\prec$ .

*Remark 6.21.* The whole computation in this section depends on the choice of the  $h_F$ -lines  $\ell_1, \dots, \ell_{22}$  whose classes form a basis of  $\text{NS}(X)$ . We have carried out the same computation on different choices of  $h_F$ -lines, and confirmed that the results are the same.

## 7. PROOF OF THEOREM 1.4

The polynomials in Table 1.1 that give a non-projective involution  $g$  of  $X_F$  are calculated by the following method. Recall that  $h'_F$  in Step 8 of the previous section is the representative vector of the  $\text{Aut}(X_F, h_F)$ -orbit  $\mathcal{V}_4 \cap \mathcal{E}_0$ . We have already calculated a birational morphism

$$\phi_{h'_F} = (\omega : \xi_0 : \xi_1 : \xi_2) : X_F \rightarrow X_{h'_F},$$

and the defining equation  $s_{h'_F}$  of  $B_{h'_F}$ . We have observed that  $s_{h'_F}$  is written as

$$s_{h'_F}(x, y, z) = \lambda^2 \mathbf{x} H {}^t \overline{\mathbf{x}},$$

where  $\lambda \in \mathbb{F}_{25}^\times$ ,  $\mathbf{x} = (x, y, z)$ ,  $\overline{\mathbf{x}} = (x^5, y^5, z^5)$  and  $H$  satisfies  $H = {}^t \overline{H}$ . We search for  $M \in \text{GL}_3(\mathbb{F}_{25})$  such that  $H = M {}^t \overline{M}$  (see Theorem 6.11), and put

$$\omega' := \lambda^{-1} \omega, \quad (\xi'_0, \xi'_1, \xi'_2) := (\xi_0, \xi_1, \xi_2) M.$$

Then the polynomials  $\omega', \xi'_0, \xi'_1, \xi'_2$  satisfy

$$\omega'^2 = \xi_0'^6 + \xi_1'^6 + \xi_2'^6.$$

Hence the rational map from  $X_F$  to  $\mathbb{P}(3, 1, 1, 1)$  given by  $(\omega' : \xi'_0 : \xi'_1 : \xi'_2)$  defines an automorphism  $\gamma$  of  $X_F$ . We choose  $h_F$ -lines  $\ell_{i_1}, \dots, \ell_{i_{22}}$  such that  $[\ell_{i_1}], \dots, [\ell_{i_{22}}]$  span  $\text{NS}(X) \otimes \mathbb{Q}$ , and that none of  $i_1, \dots, i_{22}$  is contained in the set  $J$  of indices in the expression (5.1) for  $h'_F$  that was used in the calculation of  $\phi_{h'_F}$ . Then we can calculate the images  $\ell_{i_\nu}'$  of  $\ell_{i_\nu}$  by  $\gamma$  using the parametric representations of  $\ell_{i_\nu}$  and the polynomials  $(\omega' : \xi'_0 : \xi'_1 : \xi'_2)$ . Computing the intersection numbers of  $\ell_{i_\nu}'$  with  $\ell_1, \dots, \ell_{22}$ , we calculate the action of  $\gamma$  on  $\text{NS}(X)$ . Let  $v \mapsto v\Gamma$  denote the matrix representation of this action. We then search for  $\tau \in \text{Aut}(X, h_F)$  such that its action on  $X_F$  is given by

$$w \mapsto \sigma w, \quad (x, y, z) \mapsto (x, y, z) T_\tau,$$

where  $\sigma \in \mathbb{F}_{25}^\times$ ,  $T_\tau \in \text{GU}_3(\mathbb{F}_{25})$ , and its action on  $\text{NS}(X)$  is given by  $v \mapsto vN_\tau$ , where  $N_\tau$  is a matrix satisfying  $(\Gamma N_\tau)^2 = \text{Id}_{22}$ . We define  $(\omega'', \xi''_0, \xi''_1, \xi''_2)$  by

$$\omega'' := \sigma \omega', \quad (\xi''_0, \xi''_1, \xi''_2) := (\xi'_0, \xi'_1, \xi'_2) T_\tau,$$

and replace the original polynomials  $(\omega, \xi_0, \xi_1, \xi_2)$  by  $(\omega'', \xi''_0, \xi''_1, \xi''_2)$ . Then the automorphism  $X_F \rightarrow X_F$  given by  $(\omega : \xi_0 : \xi_1 : \xi_2)$  is of order 2, because its action  $v \mapsto v\Gamma N_\tau$  on  $\text{NS}(X)$  is of order 2.

*Remark 7.1.* Let  $X_F(\mathbb{F}_{25})$  denote the set of  $\mathbb{F}_{25}$ -rational points of  $X_F$ , which consists of 1,176 points. We have confirmed that the involution  $g$  induces a permutation of order 2 on  $X_F(\mathbb{F}_{25})$ .

8. THE LIST OF PROJECTIVE MODELS  $\mathcal{E}_0, \dots, \mathcal{E}_{64}$ 

Here is the complete list of the projective models of degree 2 whose polarizations are located in  $\mathcal{B}_5 \subset \text{NS}(X)$ .

$$\mathcal{E}_0 = \overline{\mathcal{E}}_0: \quad \text{RT} = 0: \quad |\text{aut}| = 378000: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 0, 0, 126]:$$

$$N = 13051: \quad \text{stabs} \quad [756000]_2, [720]_4, [63]_5:$$

$$h = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]:$$

$$x^6 + y^6 + 1$$

$$\mathcal{E}_1 = \overline{\mathcal{E}}_1: \quad \text{RT} = 6A_1: \quad |\text{aut}| = 12: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 12, 0, 30, 18]:$$

$$N = 5607000: \quad \text{stabs} \quad [3, 12]_4, [1, 1, 1, 1, 1, 1, 2, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1]:$$

$$x^6 + 3x^5y + x^4y^2 + 2x^3y^3 + y^6 + 3x^4 + 3x^2y^2 + xy^3 + 3xy + 2y^2 + 4$$

$$\mathcal{E}_2 = \overline{\mathcal{E}}_2: \quad \text{RT} = 7A_1: \quad |\text{aut}| = 6: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 15, 0, 24, 13]:$$

$$N = 6678000: \quad \text{stabs} \quad [2]_4, [1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 6]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0]:$$

$$x^6 + 2x^4y^2 + x^2y^4 + x^2y^3 + 2y^5 + x^4 + 2y^4 + 2x^2y + 2y^3 + 3y^2 + 3y + 2$$

$$\mathcal{E}_3 = \overline{\mathcal{E}}_3: \quad \text{RT} = 3A_1 + 2A_2: \quad |\text{aut}| = 6: \quad \text{spl} = [0, 0, 0, 1, 0, 6, 0, 12, 15, 16]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0]:$$

$$x^6 + 3x^3y^3 + y^6 + 3x^3y + 2y^2 + 2$$

$$\mathcal{E}_4 = \overline{\mathcal{E}}_4: \quad \text{RT} = 8A_1: \quad |\text{aut}| = 8: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 16, 0, 24, 4]:$$

$$N = 2457000: \quad \text{stabs} \quad [4]_4, [1, 2, 2, 2, 2]_5:$$

$$h = [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0]:$$

$$x^6 + 3x^4y^2 + x^2y^4 + 4x^2y^3 + 4y^5 + x^4 + 2x^2y^2 + 3y^4 + 2x^2y + 4x^2 + y^2 + 4y$$

$$\mathcal{E}_5 = \overline{\mathcal{E}}_5: \quad \text{RT} = 8A_1: \quad |\text{aut}| = 4: \quad \text{spl} = [0, 0, 0, 5, 0, 0, 12, 0, 20, 8]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 2, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0]:$$

$$x^4y^2 + x^2y^4 + 2x^4 + 4x^2y^2 + y^4 + x^2 + 4y^2 + 4$$

$$\mathcal{E}_6 = \overline{\mathcal{E}}_6: \quad \text{RT} = 6A_1 + A_2: \quad |\text{aut}| = 6: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 9, 0, 12, 14]:$$

$$N = 1512000: \quad \text{stabs} \quad [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0]:$$

$$x^6 + 4x^4y^2 + 2x^2y^4 + 2x^2y + y^3 + 4$$

$$\mathcal{E}_7 = \overline{\mathcal{E}}_7: \quad \text{RT} = 6A_1 + A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 2, 0, 4, 9, 4, 13, 12]:$$

$$N = 4914000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 1, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1] :$$

$$\sqrt{2}x^3y^3 + (1 + 3\sqrt{2})x^2y^4 + x^4 + (2 + 2\sqrt{2})x^3y + (1 + 4\sqrt{2})x^2y^2 + xy^3 + (2 + 2\sqrt{2})y^4 + \sqrt{2}x^2 + (1 + 3\sqrt{2})xy$$

$$\mathcal{E}_8 = \overline{\mathcal{E}}_8: \quad \text{RT} = 6A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 1, 0, 4, 9, 4, 20, 6]:$$

$$N = 9828000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1] :$$

$$x^6 + 2x^5y + x^4y^2 + 3x^5 + 2xy^4 + x^3y + 3x^2y^2 + 4xy^2 + y^3 + 3y^2 + 3x + 3y$$

$$\mathcal{E}_9 = \overline{\mathcal{E}}_{10}: \quad \text{RT} = 4A_1 + 2A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 0, 0, 6, 5, 10, 13, 9]:$$

$$N = 4158000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1] :$$

$$x^5y + (2 + \sqrt{2})x^4y^2 + (1 + 4\sqrt{2})x^3y^3 + (3 + \sqrt{2})x^2y^4 + (2 + 4\sqrt{2})xy^5 + (2 + \sqrt{2})y^6 + (2 + 3\sqrt{2})x^4 + (1 + 4\sqrt{2})x^3y + (3 + \sqrt{2})y^4 + (1 + 4\sqrt{2})x^2 + (3 + \sqrt{2})xy + 3y^2 + 2 + 3\sqrt{2}$$

$$\mathcal{E}_{11} = \overline{\mathcal{E}}_{11}: \quad \text{RT} = 9A_1: \quad |\text{aut}| = 54: \quad \text{spl} = [0, 0, 0, 9, 0, 0, 0, 0, 27, 0]:$$

$$N = 84000: \quad \text{stabs} \quad [9]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, -1, 0, 0, 0] :$$

$$x^6 + 4x^3y^3 + 4y^6 + x^4 + 4xy^3 + 3x^2 + 4$$

$$\mathcal{E}_{12} = \overline{\mathcal{E}}_{12}: \quad \text{RT} = 9A_1: \quad |\text{aut}| = 9: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 18, 0, 18, 3]:$$

$$N = 1596000: \quad \text{stabs} \quad [9]_4, [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] :$$

$$4x^4y^2 + 3x^2y^4 + 4y^6 + x^5 + 3x^3y^2 + 2xy^4 + x^4 + 2x^2y^2 + 4xy^3 + 2xy^2 + 4y^3 + 4x^2 + 2xy + 1$$

$$\mathcal{E}_{13} = \overline{\mathcal{E}}_{14}: \quad \text{RT} = 9A_1: \quad |\text{aut}| = 6: \quad \text{spl} = [0, 0, 0, 6, 0, 0, 12, 0, 15, 5]:$$

$$N = 882000: \quad \text{stabs} \quad [1, 6]_5:$$

$$h = [0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1] :$$

$$\sqrt{2}x^5y + 2x^4y^2 + (3 + 2\sqrt{2})x^3y^3 + (4 + 2\sqrt{2})x^2y^4 + (4 + 4\sqrt{2})xy^5 + \sqrt{2}y^6 + (1 + \sqrt{2})x^4 + (4 + 3\sqrt{2})x^3y + (1 + 4\sqrt{2})x^2y^2 + (1 + 4\sqrt{2})y^4 + (3 + 3\sqrt{2})x^2 + (1 + \sqrt{2})xy + (3 + 4\sqrt{2})y^2 + 1 + \sqrt{2}$$

$$\mathcal{E}_{15} = \overline{\mathcal{E}}_{16}: \quad \text{RT} = 9A_1: \quad |\text{aut}| = 3: \quad \text{spl} = [0, 0, 0, 3, 0, 0, 18, 0, 12, 6]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 3, 3, 3]_5:$$

$$h = [0, -1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0] :$$

$$(2 + 2\sqrt{2})x^2y^4 + x^4y + (4 + 4\sqrt{2})x^3y^2 + (1 + \sqrt{2})x^2y^3 + (2 + 4\sqrt{2})xy^4 + (1 + \sqrt{2})x^4 + (1 + 2\sqrt{2})x^3y + (2 + 3\sqrt{2})xy^3 + (2 + 4\sqrt{2})x^2y + (2 + \sqrt{2})xy^2 + (2 + \sqrt{2})xy + 2y^2$$

$$\mathcal{E}_{17} = \overline{\mathcal{E}}_{17}: \quad \text{RT} = 9A_1: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 5, 0, 0, 12, 0, 17, 4]:$$

$$N = 3402000: \quad \text{stabs} \quad [1, 1, 1, 2, 2, 2]_5:$$

$$h = [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1] :$$

$$x^5y + 2x^4y^2 + 4x^3y^3 + 2x^2y^4 + 4xy^5 + 3y^6 + 2x^2y^2 + 2x^2 + xy$$

$$\mathcal{E}_{18} = \overline{\mathcal{E}}_{19}: \quad \text{RT} = 7A_1 + A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 1, 0, 4, 13, 3, 12, 4]:$$

$$N = 3024000: \quad \text{stabs} \quad [1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0] :$$

$$\sqrt{2}x^4y^2 + (1 + 2\sqrt{2})x^3y^3 + (3 + 4\sqrt{2})x^2y^4 + 3\sqrt{2}xy^5 + (2 + 2\sqrt{2})x^4 + \sqrt{2}x^3y + 4x^2y^2 + 3\sqrt{2}xy^3 + (2 + 2\sqrt{2})y^4 + (1 + \sqrt{2})x^2 + 4\sqrt{2}xy + (1 + \sqrt{2})y^2 + 2 + 2\sqrt{2}$$

$$\mathcal{E}_{20} = \overline{\mathcal{E}}_{21}: \quad \text{RT} = 7A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 3, 0, 4, 8, 3, 13, 6]:$$

$$N = 5292000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1] :$$

$$2\sqrt{2}x^3y^3 + (3 + \sqrt{2})x^2y^4 + x^4y + (4 + 2\sqrt{2})x^3y^2 + (3 + 4\sqrt{2})x^2y^3 + (4 + 4\sqrt{2})xy^4 + x^4 + 3\sqrt{2}x^2y^2 + 3\sqrt{2}xy^3 + 4y^4 + \sqrt{2}x^3 + 2\sqrt{2}x^2y + \sqrt{2}xy^2 + (2 + 2\sqrt{2})y^3 + 3x^2 + (3 + 2\sqrt{2})xy + (2 + 3\sqrt{2})y^2$$

$$\mathcal{E}_{22} = \overline{\mathcal{E}}_{23}: \quad \text{RT} = 7A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 2, 0, 4, 9, 3, 16, 4]:$$

$$N = 5292000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1] :$$

$$x^3y^3 + (1 + 3\sqrt{2})x^2y^4 + x^4y + (3 + 2\sqrt{2})x^3y^2 + 3\sqrt{2}x^2y^3 + (2 + 4\sqrt{2})xy^4 + \sqrt{2}x^4 + (2 + 4\sqrt{2})x^3y + 4xy^3 + (1 + 3\sqrt{2})y^4 + (2 + \sqrt{2})x^3 + (3 + 3\sqrt{2})x^2y + \sqrt{2}y^3 + (4 + 2\sqrt{2})x^2 + 4\sqrt{2}xy + (1 + 4\sqrt{2})y^2$$

$$\mathcal{E}_{24} = \overline{\mathcal{E}}_{24}: \quad \text{RT} = 5A_1 + 2A_2: \quad |\text{aut}| = 8: \quad \text{spl} = [0, 0, 0, 2, 0, 10, 0, 0, 16, 8]:$$

$$N = 378000: \quad \text{stabs} \quad [2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0] :$$

$$x^3y^3 + x^4 + x^2y^2 + y^4 + xy$$

$$\mathcal{E}_{25} = \overline{\mathcal{E}}_{26}: \quad \text{RT} = 5A_1 + 2A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 0, 0, 6, 9, 8, 6, 8]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 2, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1] :$$

$$x^2y^4 + x^4y + (1 + \sqrt{2})x^3y^2 + (3 + 4\sqrt{2})x^2y^3 + (3 + 2\sqrt{2})xy^4 + (1 + \sqrt{2})x^3y + (1 + 2\sqrt{2})x^2y^2 + (3 + \sqrt{2})xy^3 + (1 + 4\sqrt{2})x^2y + (1 + 2\sqrt{2})xy^2 + 3x^2 + 4\sqrt{2}xy + (1 + 4\sqrt{2})y^2$$



$$\mathcal{E}_{27} = \overline{\mathcal{E}}_{27}: \quad \text{RT} = 5A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 1, 0, 8, 4, 4, 13, 6]:$$

$$N = 3780000: \quad \text{stabs} \quad [1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1] :$$

$$x^6 + 3x^4y^2 + x^2y^4 + x^3y^2 + 3x^2y^3 + xy^4 + 2x^3y + 3xy^3 + 4x^3 + 3x^2y + 4xy^2 + 4y^2$$

$$\mathcal{E}_{28} = \overline{\mathcal{E}}_{29}: \quad \text{RT} = 5A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 1, 0, 6, 4, 8, 14, 4]:$$

$$N = 4536000: \quad \text{stabs} \quad [1, 1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1] :$$

$$x^4y^2 + (2 + 2\sqrt{2})x^3y^3 + (3 + 2\sqrt{2})x^2y^4 + (1 + \sqrt{2})x^4y + 2\sqrt{2}x^3y^2 + (2 + \sqrt{2})xy^4 + (2 + 3\sqrt{2})x^4 + 4x^2y^2 + (1 + 3\sqrt{2})y^4 + (3 + 4\sqrt{2})x^3 + 4\sqrt{2}xy^2 + (1 + \sqrt{2})y^3 + (4 + 2\sqrt{2})x^2 + (3 + 3\sqrt{2})xy + (1 + 2\sqrt{2})y^2$$

$$\mathcal{E}_{30} = \overline{\mathcal{E}}_{31}: \quad \text{RT} = 3A_1 + 3A_2: \quad |\text{aut}| = 3: \quad \text{spl} = [0, 0, 0, 0, 0, 6, 3, 15, 6, 6]:$$

$$N = 1260000: \quad \text{stabs} \quad [1, 3, 3]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1] :$$

$$x^4y^2 + (1 + \sqrt{2})x^3y^3 + (2 + 3\sqrt{2})x^2y^4 + x^4y + 4x^3y^2 + (3 + 3\sqrt{2})x^2y^3 + 4\sqrt{2}xy^4 + 4x^4 + (2 + 3\sqrt{2})x^3y + x^2y^2 + (4 + 2\sqrt{2})y^4 + (3 + 2\sqrt{2})x^3 + (4 + 3\sqrt{2})x^2y + (4 + 4\sqrt{2})xy^2 + (2 + 4\sqrt{2})y^3 + x^2 + \sqrt{2}xy + 3y^2$$

$$\mathcal{E}_{32} = \overline{\mathcal{E}}_{32}: \quad \text{RT} = 10A_1: \quad |\text{aut}| = 20: \quad \text{spl} = [0, 0, 0, 0, 0, 0, 20, 0, 10, 1]:$$

$$N = 226800: \quad \text{stabs} \quad [20]_4, [4]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1] :$$

$$x^6 + 2x^4y + y^5 + 4x^2y^2 + y^3 + 4x^2 + 4y$$

$$\mathcal{E}_{33} = \overline{\mathcal{E}}_{33}: \quad \text{RT} = 10A_1: \quad |\text{aut}| = 4: \quad \text{spl} = [0, 0, 0, 6, 0, 0, 16, 0, 4, 6]:$$

$$N = 756000: \quad \text{stabs} \quad [2, 2]_5:$$

$$h = [0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0] :$$

$$x^6 + x^4y^2 + 3x^3y^3 + 3x^2y^4 + 2y^6 + x^2y^2 + 4xy + 4$$

$$\mathcal{E}_{34} = \overline{\mathcal{E}}_{35}: \quad \text{RT} = 10A_1: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 4, 0, 0, 17, 0, 9, 3]:$$

$$N = 1890000: \quad \text{stabs} \quad [1, 1, 2]_5:$$

$$h = [0, -1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0] :$$

$$x^5y + x^4y^2 + 3x^3y^3 + (4 + \sqrt{2})x^2y^4 + (1 + \sqrt{2})xy^5 + 4\sqrt{2}y^6 + 2x^4 + 4x^3y + (4 + 4\sqrt{2})xy^3 + (2 + 2\sqrt{2})y^4 + x^2 + (1 + 4\sqrt{2})y^2 + 2$$

$$\mathcal{E}_{36} = \overline{\mathcal{E}}_{36}: \quad \text{RT} = 8A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 3, 0, 4, 11, 2, 7, 4]:$$

$$N = 3780000: \quad \text{stabs} \quad [1, 1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1] :$$

$$x^5y + 4x^2y^4 + x^5 + 3x^4y + 2x^2y^3 + 3x^4 + 2y^4 + 2xy^2 + 2y^3 + 2x^2 + 3xy + 4y$$

$$\mathcal{E}_{37} = \overline{\mathcal{E}}_{37}: \quad \text{RT} = 8A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 2, 0, 4, 13, 2, 6, 6]:$$

$$N = 3024000: \quad \text{stabs} \quad [1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1] :$$

$$x^4 y^2 + 4 x^3 y^3 + 4 x^2 y^4 + 3 x y^4 + y^5 + 4 x y^3 + 4 x^3 + 4 x^2 y + 4 x^2 + x y + 3 y^2 + 3 x + 3 y$$

$$\mathcal{E}_{38} = \overline{\mathcal{E}}_{39}: \quad \text{RT} = 8A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 2, 0, 4, 12, 2, 10, 2]:$$

$$N = 3024000: \quad \text{stabs} \quad [1, 1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1] :$$

$$(1 + 4\sqrt{2}) x^2 y^4 + x^4 y + (1 + \sqrt{2}) x^3 y^2 + 3 x^2 y^3 + (2 + \sqrt{2}) x y^4 + x^4 + (2 + 2\sqrt{2}) x^3 y + 3 x^2 y^2 + \sqrt{2} y^4 + 4 \sqrt{2} x^3 + (2 + 3\sqrt{2}) x^2 y + y^3 + 3 x^2 + (2 + 4\sqrt{2}) x y + 3 y^2$$

$$\mathcal{E}_{40} = \overline{\mathcal{E}}_{41}: \quad \text{RT} = 6A_1 + 2A_2: \quad |\text{aut}| = 6: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 6, 6, 6, 5]:$$

$$N = 378000: \quad \text{stabs} \quad [2]_5:$$

$$h = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0] :$$

$$\sqrt{2} x^6 + (1 + \sqrt{2}) x^5 y + (1 + 4\sqrt{2}) x^3 y^3 + \sqrt{2} x^2 y^4 + 2 \sqrt{2} x y^5 + (3 + \sqrt{2}) y^6 + (4 + 3\sqrt{2}) x^4 + 3 x^3 y + (2 + \sqrt{2}) x^2 y^2 + (4 + 4\sqrt{2}) x y^3 + (3 + 3\sqrt{2}) y^4 + (2 + \sqrt{2}) x^2 + (1 + 4\sqrt{2}) x y + \sqrt{2} y^2 + 4$$

$$\mathcal{E}_{42} = \overline{\mathcal{E}}_{43}: \quad \text{RT} = 6A_1 + 2A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 5, 6, 10, 1]:$$

$$N = 1512000: \quad \text{stabs} \quad [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1] :$$

$$x^4 y^2 + (2 + 3\sqrt{2}) x^3 y^3 + (3 + 3\sqrt{2}) x^2 y^4 + x^4 y + (3 + 3\sqrt{2}) x^2 y^3 + 3 \sqrt{2} x y^4 + 2 \sqrt{2} x^4 + (3 + 4\sqrt{2}) x^2 y^2 + 2 \sqrt{2} x y^3 + (4 + \sqrt{2}) y^4 + (3 + 3\sqrt{2}) x^3 + (4 + 3\sqrt{2}) y^3 + (4 + 2\sqrt{2}) x^2 + 4 \sqrt{2} x y + (3 + 2\sqrt{2}) y^2$$

$$\mathcal{E}_{44} = \overline{\mathcal{E}}_{45}: \quad \text{RT} = 6A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 1, 0, 6, 7, 6, 9, 3]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1] :$$

$$2 x^3 y^3 + 3 \sqrt{2} x^2 y^4 + (4 + 2\sqrt{2}) x^3 y^2 + (3 + \sqrt{2}) x^2 y^3 + (1 + 2\sqrt{2}) x y^4 + x^4 + (3 + \sqrt{2}) x^3 y + 3 x^2 y^2 + 3 y^4 + (1 + 4\sqrt{2}) x^3 + (1 + 3\sqrt{2}) x^2 y + 4 x y^2 + (2 + 4\sqrt{2}) y^3 + (3 + \sqrt{2}) x^2 + (1 + \sqrt{2}) x y + (3 + 3\sqrt{2}) y^2$$

$$\mathcal{E}_{46} = \overline{\mathcal{E}}_{46}: \quad \text{RT} = 4A_1 + 3A_2: \quad |\text{aut}| = 3: \quad \text{spl} = [0, 0, 0, 1, 0, 6, 3, 12, 7, 0]:$$

$$N = 756000: \quad \text{stabs} \quad [1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1] :$$

$$x^6 + 3 x^3 y^3 + 4 x^4 y + x y^4 + 3 x^2 y^2 + 4 x^3 + 3 x y + 4$$

$$\mathcal{E}_{47} = \overline{\mathcal{E}}_{47}: \quad \text{RT} = 4A_1 + 3A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 0, 0, 8, 4, 8, 4, 5]:$$

$$N = 1134000: \quad \text{stabs} \quad [1, 2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0] :$$

$$x^6 + 3x^4y^2 + 4x^2y^4 + 2y^6 + 4x^2y^3 + 2x^4 + 3x^2y^2 + 4x^2y + y^3 + 3x^2$$

$$\mathcal{E}_{48} = \overline{\mathcal{E}}_{48}: \quad \text{RT} = 4A_1 + 3A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 0, 0, 8, 5, 8, 5, 4]:$$

$$N = 2268000: \quad \text{stabs} \quad [1, 1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1] :$$

$$2x^4y^2 + x^5 + 2x^2y^3 + 4xy^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 2xy^2 + 3x^2 + 2xy + 2y^2$$

$$\mathcal{E}_{49} = \overline{\mathcal{E}}_{49}: \quad \text{RT} = 11A_1: \quad |\text{aut}| = 4: \quad \text{spl} = [0, 0, 0, 8, 0, 0, 8, 0, 10, 0]:$$

$$N = 378000: \quad \text{stabs} \quad [2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0] :$$

$$x^6 + x^4y^2 + 4x^2y^4 + 3x^5 + 3xy^4 + x^2y^2 + 2y^4 + x^3 + 4y^2 + 2x + 2$$

$$\mathcal{E}_{50} = \overline{\mathcal{E}}_{51}: \quad \text{RT} = 9A_1 + A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 4, 0, 4, 9, 1, 7, 2]:$$

$$N = 1512000: \quad \text{stabs} \quad [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1] :$$

$$(4 + \sqrt{2})x^3y^3 + (4 + 2\sqrt{2})x^2y^4 + x^4y + 4xy^4 + \sqrt{2}x^4 + (3 + 3\sqrt{2})x^2y^2 + 4xy^3 + (4 + 2\sqrt{2})y^4 + (2 + 3\sqrt{2})x^3 + (4 + 4\sqrt{2})x^2y + (4 + 3\sqrt{2})y^3 + (1 + 2\sqrt{2})x^2 + 3\sqrt{2}xy + (2 + 3\sqrt{2})y^2$$

$$\mathcal{E}_{52} = \overline{\mathcal{E}}_{52}: \quad \text{RT} = 7A_1 + 2A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 4, 0, 6, 2, 4, 8, 2]:$$

$$N = 378000: \quad \text{stabs} \quad [2]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0] :$$

$$x^6 + x^5y + 2x^4y^2 + x^2y^4 + 3y^6 + x^4 + x^2y^2 + xy^3 + 4xy + y^2 + 3$$

$$\mathcal{E}_{53} = \overline{\mathcal{E}}_{54}: \quad \text{RT} = 7A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 7, 4, 7, 0]:$$

$$N = 1512000: \quad \text{stabs} \quad [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1] :$$

$$(2 + 2\sqrt{2})x^2y^4 + x^4y + (4 + \sqrt{2})x^3y^2 + (2 + 2\sqrt{2})x^2y^3 + 4xy^4 + 3x^4 + (3 + 2\sqrt{2})x^3y + \sqrt{2}x^2y^2 + (3 + 4\sqrt{2})xy^3 + (2 + 2\sqrt{2})y^4 + (3 + 4\sqrt{2})x^2y + (2 + 3\sqrt{2})xy^2 + (2 + 3\sqrt{2})y^3 + \sqrt{2}xy + 4y^2$$

$$\mathcal{E}_{55} = \overline{\mathcal{E}}_{56}: \quad \text{RT} = 7A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 6, 4, 6, 1]:$$

$$N = 1512000: \quad \text{stabs} \quad [1, 1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0] :$$

$$(3 + 2\sqrt{2})x^2y^4 + x^4y + x^3y^2 + (2 + \sqrt{2})x^2y^3 + (2 + 4\sqrt{2})xy^4 + \sqrt{2}x^4 + (3 + 4\sqrt{2})x^3y + (2 + 4\sqrt{2})xy^3 + 4\sqrt{2}y^4 + \sqrt{2}x^3 + (1 + 4\sqrt{2})x^2y + (4 + \sqrt{2})y^3 + 4x^2 + 4\sqrt{2}xy + 2y^2$$

$$\mathcal{E}_{57} = \overline{\mathcal{E}}_{58}: \quad \text{RT} = 5A_1 + 3A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 1, 0, 6, 4, 9, 4, 1]:$$

$$N = 756000: \quad \text{stabs} \quad [1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1] :$$

$$\sqrt{2}x^4y^2 + (4 + 2\sqrt{2})x^3y^3 + 4\sqrt{2}x^2y^4 + (3 + \sqrt{2})xy^5 + 4\sqrt{2}y^6 + (1 + 4\sqrt{2})x^4 + (3 + \sqrt{2})x^3y + 2\sqrt{2}x^2y^2 + (1 + \sqrt{2})y^4 + (3 + 2\sqrt{2})x^2 + (2 + 4\sqrt{2})xy + (3 + \sqrt{2})y^2 + 1 + 4\sqrt{2}$$

$$\mathcal{E}_{59} = \overline{\mathcal{E}}_{60}: \quad \text{RT} = 8A_1 + 2A_2: \quad |\text{aut}| = 2: \quad \text{spl} = [0, 0, 0, 2, 0, 6, 7, 2, 5, 0]:$$

$$N = 378000: \quad \text{stabs} \quad [2]_5:$$

$$h = [0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0] :$$

$$x^5y + (1 + \sqrt{2})x^4y^2 + 2x^2y^4 + (2 + \sqrt{2})xy^5 + (4 + 3\sqrt{2})y^6 + 3x^4 + (4 + 4\sqrt{2})x^3y + (1 + 3\sqrt{2})x^2y^2 + (3 + 3\sqrt{2})xy^3 + 4\sqrt{2}y^4 + 4x^2 + \sqrt{2}xy + (3 + 3\sqrt{2})y^2 + 3$$

$$\mathcal{E}_{61} = \overline{\mathcal{E}}_{62}: \quad \text{RT} = 8A_1 + 2A_2: \quad |\text{aut}| = 1: \quad \text{spl} = [0, 0, 0, 3, 0, 6, 7, 2, 3, 1]:$$

$$N = 756000: \quad \text{stabs} \quad [1]_5:$$

$$h = [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1] :$$

$$x^3y^3 + (3 + 4\sqrt{2})x^2y^4 + 2\sqrt{2}x^3y^2 + 2x^2y^3 + 2xy^4 + x^4 + (4 + 2\sqrt{2})x^3y + (1 + \sqrt{2})x^2y^2 + (3 + 2\sqrt{2})y^4 + (3 + \sqrt{2})x^3 + (3 + 4\sqrt{2})x^2y + 4\sqrt{2}y^3 + (2 + 4\sqrt{2})x^2 + (3 + 2\sqrt{2})xy + (4 + 4\sqrt{2})y^2$$

$$\mathcal{E}_{63} = \overline{\mathcal{E}}_{63}: \quad \text{RT} = 6A_1 + 3A_2: \quad |\text{aut}| = 3: \quad \text{spl} = [0, 0, 0, 3, 0, 6, 3, 6, 3, 0]:$$

$$N = 252000: \quad \text{stabs} \quad [3]_5:$$

$$h = [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0] :$$

$$x^4y^2 + x^4y + x^3y^2 + 2y^5 + 2x^4 + 4x^2y^2 + 3xy^3 + 4y^4 + 4x^3 + 2xy^2 + y^3 + 2x^2 + y^2$$

$$\mathcal{E}_{64} = \overline{\mathcal{E}}_{64}: \quad \text{RT} = 6A_1 + 3A_2: \quad |\text{aut}| = 3: \quad \text{spl} = [0, 0, 0, 0, 0, 6, 9, 6, 0, 1]:$$

$$N = 252000: \quad \text{stabs} \quad [3]_5:$$

$$h = [0, -1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1] :$$

$$x^4y^2 + x^3y^3 + x^2y^4 + 3x^3y^2 + x^2y^3 + 3x^3y + x^2y^2 + 2x^3 + 2x^2y + 3y^3 + 2x^2 + 3xy + 4y + 4$$

## 9. EXAMPLES WITH MILNOR NUMBER $\geq 20$

In [14], we have shown that every supersingular  $K3$  surface in characteristic 5 with Artin invariant  $\leq 3$  is isomorphic to the minimal resolution  $X_f$  of the double cover  $Y_f \rightarrow \mathbb{P}^2$  of  $\mathbb{P}^2$  branching along a curve  $B_f$  of degree 6 whose affine part is defined by

$$y^5 - f(x) = 0,$$

$f_i$	$\text{Sing}(B_{f_i})$
$x^3(x-1)^3$	$2E_8 + A_4$
$x^3(x-1)^2$	$A_9 + E_8 + A_4$
$x^3(x-1)^2(x-2)$	$E_8 + 3A_4$
$x^2(x-1)^2$	$A_9 + 3A_4$
$x^2(x-1)^2(x+2\omega+3)$	$A_9 + 3A_4$
$x^2(x-1)^2(x^2-x+2)$	$5A_4$
$x^2(x-1)^2(x+1)(x+3)$	$5A_4$
$x^2(x-1)^2(x^2-\omega x+\omega)$	$5A_4$
$x^2(x-1)^2(x^2-\bar{\omega}x+\bar{\omega})$	$5A_4$ .

TABLE 9.1. The polynomials  $f_i$ 

where  $f$  is a polynomial of degree  $\leq 6$ . (When  $\deg f < 6$ , the sextic curve  $B_f$  contains the line at infinity.) Investigating the singular points of  $B_f$  and the splitting curves of degree  $\leq 3$  of  $Y_f \rightarrow \mathbb{P}^2$ , we obtain the following:

**Theorem 9.1.** *Let  $\omega \in \mathbb{F}_{25}$  be a root of  $\omega^2 + \omega + 1 = 0$ , and let  $\bar{\omega}$  be  $\omega^5$ . Then  $X_f$  is a supersingular  $K3$  with Artin invariant 1 if and only if  $B_f \subset \mathbb{P}^2$  is projectively isomorphic to the curve corresponding to one of the  $f_i$  in Table 9.1. These nine sextic curves  $B_{f_i}$  are not projectively isomorphic to each other.*  $\square$

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